Sums of Things

The "sigma notation" for sums is important and handy. We're assuming you're familiar with it. If you need to learn (or review) it, look at Appendix F in the back of the textbook. If you're confused about it, please talk to your instructor or TA.

As you saw in Section 5.1, there are certain sum formulas that are handy for figuring out the exact value of certain areas. These are the formulas for the "sum of the k^{th} powers of the numbers 1,2,...,n. These notes talk about those formulas (listed in Appendix F). They include what you are required to know, and some additional optional information for those who are interested.

We'll use the notation S_k for the sum of the k^{th} powers, so S_k is short for

$$S_k = 1^k + 2^k + \ ... \ + n^k.$$

Of course, this is a "formula" for S_k, but it doesn't help you compute – it doesn't tell you how to find the exact value, say, of $S_3 = 1^3 + 2^3 + \ ... \ + 15^3$. We'd like to get what's called a "closed formula" for S_k, that is, one without the annoying " ... " in it.

For $S_0 = 1^0 + 2^0 + \ ... \ + n^0$, this is easy: since there are n terms, each $= 1$, we get

$$S_0 = 1 + 1 + ... + 1 \ = 1 \cdot n \ = n$$

For S_1, it's already harder. Here's a slick way of finding a closed formula for S_1:

Write down S_1 twice, in two different orders:

$$\begin{array}{lcccccccl} S_1 = & 1 & + & 2 & + & 3 & + ... + & (n-1) + & n, \text{ and} \\ S_1 = & n & + & (n-1) & + & (n-2) & + ... + & 2 & + \ 1 \quad \text{and then add to get:} \\ \hline 2S_1 = & (n+1) & + & (n+1) & + & (n+1) & + ... + & (n+1) + & (n+1). \end{array}$$

Since there are n terms on the right, each equal to $(n+1)$, we get

$$2S_1 = n(n+1), \text{ so}$$

$$S_1 = \frac{n(n+1)}{2}$$

This is a "usable" closed formula: for example, $1 + 2 + 3 + ... + 15 \ = \frac{15(16)}{2} = 120$.

In Appendix F, several such formulas are given. Here's a list:

$$\begin{aligned} S_0 &= 1^0 + 2^0 + \ ... \ + \ n^0 = n \\ S_1 &= 1^1 + 2^1 + ... \ + n^1 = \frac{n(n+1)}{2} \\ S_2 &= 1^2 + 2^2 + ... \ + n^2 = \frac{n(n+1)(2n+1)}{6} \\ S_3 &= [\frac{n(n+1)}{2}]^2 \quad \text{(Curious observation: } S_3 = [S_1]^2) \end{aligned}$$

You are responsible for knowing the formulas for S_0, S_1, and S_2. If S_3 is needed (say on a test) it will be given to you.

Optional Material

Where do these formulas come from? As illustrated above, it's easy to discover formulas for S_0 and S_1. Also (if you know how to do proofs by induction), it's not too hard to prove the formula for S_2 and, with some work, even S_3. But even then, to prove these formulas by induction, you need to guess (or be told) first what the formula is that you're trying to prove. How did someone figure out (or guess) the formula for S_3 in the first place?

There is a way to get the formulas for each S_k once you know the previous ones. (We say that this is a "recursive" way to find S_k. It allows you to find S_3, for example, once you know S_0, S_1 and S_2.) The bigger the k value is, the harder the algebra, but it always works.

Here's how it works (an idea I first read in George Polya's book, *Mathematical Discovery*):

We can see right away the $S_0 = n$. How do we use S_0 to find S_1?

For any j, we know that $(j+1)^2 - j^2 = 2j+1$. We write this down for each $j = 1, 2, ..., n$

$$
\begin{aligned}
2^2 - 1^2 &= 2(1) + 1 \\
3^2 - 2^2 &= 2(2) + 1 \\
4^2 - 3^2 &= 2(3) + 1 \\
&\dots \\
(n+1)^2 - n^2 &= 2(n) + 1
\end{aligned}
$$

Adding up both sides gives (with lots of cancellations on the left-hand side)

$$
\begin{aligned}
(n+1)^2 - 1 &= 2(1) + 2(2) + ... + 2(n) + (1 + 1 + ... + 1) \\
&= 2(1 + 2 + ... + n) + n \\
&= 2S_1 + n \quad \text{so}
\end{aligned}
$$

$n^2 + 2n + 1 - 1 = 2S_1 + n$ so
$n^2 + n = 2S_1$ so
$\frac{n^2+n}{2} = \frac{n(n+1)}{2} = S_1.$

So now we know formulas for S_0 and S_1. How can we get a formula for S_2 ? It's the same idea, but a little more algebra.

For any j we know that $(j+1)^3 - j^3 = 3j^2 + 3j + 1$. We write this down for each $j = 1, 2, ..., n.$

$$
\begin{aligned}
2^3 - 1^3 &= 3(1^2) + 3(1) + 1 \\
3^3 - 2^3 &= 3(2^2) + 3(2) + 1 \\
4^3 - 3^3 &= 3(3^2) + 3(3) + 1 \\
&\dots.. \\
(n+1)^3 - n^3 &= 3(n^2) + 3(n) + 1.
\end{aligned}
$$

Add up both sides to get

$$
\begin{aligned}
(n+1)^3 - 1 &= 3(1^2 + 2^2 + ... + n^2) + 3(1 + 2 + ... + n) + (1 + 1 + .. + 1) \\
&= 3S_2 \qquad\qquad\qquad\qquad + 3S_1 \qquad\qquad\qquad + S_0, \quad \text{so}
\end{aligned}
$$

$n^3 + 3n^2 + 3n + 1 - 1 = 3S_2 + 3S_1 + S_0$. Now we solve to get S_2.
$n^3 + 3n^2 + 3n - 3S_1 - S_0 = 3S_2$. But we know S_1 and S_2, so

$$n^3 + 3n^2 + 3n - 3[\frac{n(n+1)}{2}] - n = 3S_2.$$

The right hand side simplifies as

$$\frac{2n^3+6n^2+6n-3n^2-3n-2n}{2} = \frac{2n^3+3n^2+n}{2} = \frac{n(2n^2+3n+1)}{2} = \frac{n(n+1)(2n+1)}{2} = 3S_2, \text{ so we get}$$

$$S_2 = \frac{n(n+1)(2n+1)}{6}.$$

The same idea (with more and more algebra, using the binomial theorem to expand powers like $(n+1)^k$) works each time. Here's the idea. See if you can use it to derive the formula for S_3 given above.

(Recall the formula for the "binomial coefficients" $\binom{k}{l} = \frac{k!}{l!\,(k-l)!}$)

Suppose we have figured out formulas for $S_0, S_1, S_2, \ldots, S_{k-1}$. We know (from the binomial theorem) that for any j,

$(j+1)^{k+1} - j^{k+1} = \binom{k+1}{1} j^k + \binom{k+1}{2} j^{k-1} + \binom{k+1}{3} j^{k-2} + \ldots + 1$ Then write out for $j = 1, 2, \ldots n$

$$\begin{aligned}
2^{k+1} - 1^{k+1} &= \tbinom{k+1}{1} 1^k + \tbinom{k+1}{2} 1^{k-1} + \tbinom{k+1}{3} 1^{k-2} \ldots + 1 \\
3^{k+1} - 2^{k+1} &= \tbinom{k+1}{1} 2^k + \tbinom{k+1}{2} 2^{k-1} + \tbinom{k+1}{3} 2^{k-2} \ldots + 1 \\
&\ldots\ldots \\
(n+1)^{k+1} - n^{k+1} &= \tbinom{k+1}{1} n^k + \tbinom{k+1}{2} n^{k-1} + \tbinom{k+1}{3} n^{k-2} \ldots + 1.
\end{aligned}$$

Add the columns to get

$$\begin{aligned}
(n+1)^{k+1} - n^{k+1} &= \tbinom{k+1}{1}(1^k + 2^k + \ldots + n^k) + \tbinom{k+1}{2}(1^{k-1} + 2^{k-1} + \ldots + n^{k-1}) \\
&\quad + \tbinom{k+1}{3}(1^{k-2} + 2^{k-2} + \ldots + n^{k-1}) \ldots + n \\
&= \tbinom{k+1}{1} S_k + \tbinom{k+1}{2} S_{k-1} + \tbinom{k+1}{3} S_{k-2} + \ldots + S_0.
\end{aligned}$$

Then we solve for what we want:

$$\begin{aligned}
S_k &= [(n+1)^{k+1} - n^{k+1} - \tbinom{k+1}{2} S_{k-1} - \tbinom{k+1}{3} S_{k-2} - \ldots - S_0] / \tbinom{k+1}{1} \\
&= [(n+1)^{k+1} - n^{k+1} - \tbinom{k+1}{2} S_{k-1} - \tbinom{k+1}{3} S_{k-2} - \ldots - S_0] / (k+1)
\end{aligned}$$

We are assuming that we already have formulas for $S_{k-1}, S_{k-2}, \ldots S_1, S_0$ – which we then substitute into this formula to get one complete, if complicated, formula for S_k in terms of n.