Connecting the numbers $e,\pi,i,1,0$

The five numbers in the title are arguably the most important constants in mathematics. Without proof (just manipulating series mechanically) we can find a connection between them.

You recall the "imaginary" or "complex number" symbol i from the days of studying the quadratic formula. The number i is introduced as a new "number" with the property that $i^2 = -1$, that is $i = \sqrt{-1}$. We can also consider other complex numbers like $1+3i$, $7-\frac{3}{8}i$, $-9i$, etc. In general, a complex number looks like $z = a + bi$ where a and b are real numbers. Note that

$$
\begin{array}{ll}
i & i^5 = i^4 \cdot i = i \\
i^2 = -1 & i^6 = i^4 \cdot i^2 = -1 \\
i^3 = i \cdot i^2 = i(-1) = -i & i^7 = i^4 \cdot i^3 = -i \\
i^4 = i^2 \cdot i^2 = 1 & i^8 = i^4 \cdot i^4 = 1
\end{array}
$$

etc.

It turns out that all the facts of calculus can be developed using complex numbers instead of real numbers. In particular, Taylor series can be developed and <u>it turns out</u> that exactly the same formulas for the sin, cos, and exponential functions hold true:

$$e^z = 1 + z + \tfrac{z^2}{2!} + \tfrac{z^3}{3!} + \tfrac{z^4}{4!} + \tfrac{z^5}{5!} + \ldots. \qquad = \sum_{n=0}^{\infty} \tfrac{z^n}{n!} \text{ for all } z$$

$$\sin z = z - \tfrac{z^3}{3!} + \tfrac{z^5}{5!} - \tfrac{z^7}{7!} + \ldots \qquad = \sum_{n=0}^{\infty} (-1)^n \tfrac{z^{2n+1}}{(2n+1)!} \text{ for all } z$$

$$\cos z = 1 - \tfrac{z^2}{2!} + \tfrac{z^4}{4!} - \tfrac{z^6}{6!} + \ldots \qquad = \sum_{n=0}^{\infty} (-1)^n \tfrac{z^{2n}}{(2n)!} \text{ for all } z$$

For any real number x, we can substitute the complex number $z = ix$ into these series and simplify. Watch what happens:

$$e^{ix} = 1 + (ix) + \tfrac{(ix)^2}{2!} + \tfrac{(ix)^3}{3!} + \tfrac{(ix)^4}{4!} + \tfrac{(ix)^5}{5!} + \tfrac{(ix)^6}{6!} + \tfrac{(ix)^7}{7!} - \ldots =$$

$$= 1 + ix - \tfrac{x^2}{2!} - i\tfrac{x^3}{3!} + \tfrac{x^4}{4!} + i\tfrac{x^5}{5!} - \tfrac{x^6}{6!} + - i\tfrac{x^7}{7!} + \ldots =$$

$$= (1 - \tfrac{x^2}{2!} + \tfrac{x^4}{4!} - \tfrac{x^6}{6!} + \ldots) + i(x - \tfrac{x^3}{3!} + \tfrac{x^5}{5!} - \tfrac{x^7}{7!} +) =$$

$$= \cos x + i \sin x \quad \text{(for all real } x)$$

This shows that the sin, cos and exponential functions are all intimately tied together through the complex numbers (see Math 416, *Complex Variables*). <u>In particular</u>, if we let $x = \pi$, we get $e^{i\pi} = \cos \pi + i \sin \pi = -1 + 0 = -1$. Therefore $e^{i\pi} + 1 = 0$, an equation that relates what are (arguably) the 5 most important constants in mathematics!