Proof Let $d(a,b) = \epsilon > 0$. Since a and b are points interior to A, we can choose positive radii ϵ_a and ϵ_b so that $B_{\epsilon_a}(a) \subseteq \text{int}\,A$ and $B_{\epsilon_b}(b) \subseteq \text{int}\,A$. In choosing, we may also take both ϵ_a and ϵ_b less than $\frac{\epsilon}{2}$. Let $\epsilon' = \min\{\epsilon_a, \epsilon_b\}$. Then clearly the closed balls $F_a = \{x : d(x,a) \leq \epsilon'\}$ and $F_b = \{x : d(x,b) \leq \epsilon'\}$ meet the conditions required. •

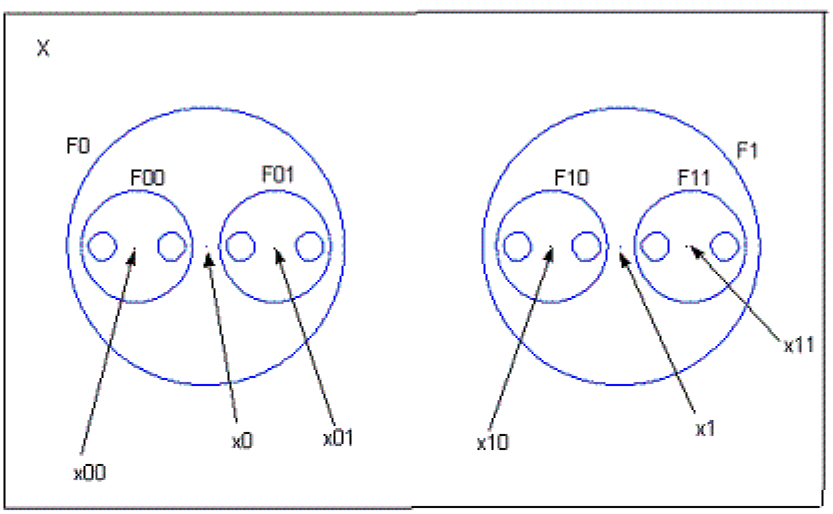
Proof of Theorem 3.6 The idea of the proof is to construct inductively c descending sequences of closed sets, each satisfying the condition 2) in the Cantor Intersection Theorem, and to do this in a way that the intersection of each sequence gives a different point in X. It then follows that $|X| \geq c$. The idea is simple although the notation gets a bit complicated. First we will give the idea of how the construction is done. The actual details of the induction step are relegated to the end of the proof.

X is nonempty and has no isolated points, so there must exist two points $x_0 \neq x_1 \in X$. Pick disjoint closed balls F_0 and F_1, centered at x_0 and x_1, each with diameter $< \frac{1}{2}$.

Since $x_0 \in \text{int}\,F_0$ and x_0 is not isolated, we can pick distinct points x_{00} and x_{01} (both $\neq x_0$) in $\text{int}\,F_0$ and use Lemma 3.5 to pick disjoint closed balls F_{00} and F_{01} (centered at x_{00} and x_{01}) and both $\subseteq \text{int}\,F_0$. We can then shrink the balls, if necessary, so that each has diameter $< \frac{1}{2^2}$.

We can repeat the same procedure with F_1 : since $x_1 \in \text{int}\,F_1$ and x_1 is not isolated, we can pick distinct points x_{10} and x_{11} (both $\neq x_1$) in $\text{int}\,F_1$ and use the Lemma 3.5 to pick disjoint closed balls F_{10} and F_{11} (centered at x_{10} and x_{11}) and both $\subseteq \text{int}\,F_1$. We can then shrink the balls, if necessary, so that each has diameter $< \frac{1}{2^2}$.

We now repeat the same construction inside each of the 4 sets $F_{00}, F_{01}, F_{10}, F_{11}$. For example, we can pick distinct points x_{000} and x_{001} (both $\neq x_{00}$) in $\text{int}\,F_{00}$ and use the Lemma 3.5 pick disjoint closed balls F_{000} and F_{001} (centered at x_{000} and x_{001}) and both $\subseteq \text{int}\,F_{00}$. We can then shrink the balls, if necessary, so that each has diameter $< \frac{1}{2^3}$. *See the figure below.*



At this stage we have the beginnings for 8 descending sequences of nonempty closed sets. In each sequence, at the n^{th} "stage," the sets have diameter $< \frac{1}{2^n}$.

$$
\begin{array}{lll}
 & & F_{000} \\
 & F_{00} \supseteq & \\
 & & F_{001} \\
F_0 \supseteq & & \\
 & & F_{010} \\
 & F_{01} \supseteq & \\
 & & F_{011}
\end{array}
$$

and

$$
\begin{array}{lll}
 & & F_{100} \\
 & F_{10} \supseteq & \\
 & & F_{101} \\
F_1 \supseteq & & \\
 & & F_{110} \\
 & F_{11} \supseteq & \\
 & & F_{111}
\end{array}
$$

We continue inductively (*see the details below*) in this way – each descending sequence splits into two new "disjoint branches" at each stage. For the next step, we make sure that we choose nonempty disjoint closed sets that descend "deeper" and that their diameters keep shrinking toward 0.

In the end, we will have one such descending sequence of closed sets for each for each $s = (n_1, n_2, ..., n_k, ...) \in \{0, 1\}^{\mathbb{N}}$ (that is, for each binary sequence).

> For example, the binary sequence $s = (0, 1, 1, 0, 0, 1...) \in \{0, 1\}^{\mathbb{N}}$ corresponds to the descending sequence of closed sets
>
> $$F_0 \supseteq F_{01} \supseteq F_{011} \supseteq F_{0110} \supseteq F_{01100} \supseteq F_{011001} \supseteq ...$$

The Cantor Intersection theorem tells us for each such s, there is an $x_s \in X$ such that

$$F_{n_1} \cap F_{n_2 n_2} \cap F_{n_1 n_2 n_3} \cap ... \cap F_{n_1 n_2 ... n_k} \cap ... = \bigcap_{k=1}^{\infty} F_{n_1 n_2 ... n_k} = \{x_s\}$$

For two different binary sequences, say $t = (m_1, m_2, ..., m_k, ...) \neq (n_1, n_2, ..., n_k, ...) = s$, there is a <u>smallest</u> k for which $m_k \neq n_k$. Then $x_s \in F_{n_1 n_2 ... n_{k-1} n_k}$ and $x_t \in F_{n_1 n_2 ... n_{k-1} m_k}$. Since these sets are disjoint, we have $x_s \neq x_t$. Thus, mapping $s \mapsto x_s$ gives a $1-1$ function from $\{0, 1\}^{\mathbb{N}}$ into$X$. We conclude that $c = 2^{\aleph_0} = |\{0, 1\}|^{\mathbb{N}} \leq |X|$.

<u>Here are the details of the formal induction step in the proof.</u>.

Induction Hypothesis: Suppose we have completed k stages – that is, for each $i = 1, ..., k$ and for each i-tuple $(n_1, ..., n_i) \in \{0, 1\}^i$ we have defined points $x_{n_1 ... n_i}$ and closed balls $F_{n_1 ... n_i}$ centered at $x_{n_1 ... n_i}$, with diam$(F_{n_1 ... n_i}) < \frac{1}{2^i}$ and so that for each $(n_1, ..., n_i)$, $F_{n_1} \supseteq F_{n_1 n_2} \supseteq ... \supseteq F_{n_1 n_2 ... n_i}$

Induction step: We must construct the sets for stage $k+1$. For each $(k+1)$-tuple $(n_1, ..., n_k, n_{k+1}) \in \{0,1\}^{k+1}$, we need to define a point $x_{n_1 ... n_k n_{k+1}}$ and a closed ball $F_{n_1 ... n_k n_{k+1}}$ in such a way that the conditions in the induction hypothesis remain true with $k+1$ replacing k.

For any $(n_1, ..., n_k)$: we have $x_{n_1,,n_k} \in \text{int}\, F_{n_1 ... n_k}$. Since $x_{n_1,,n_k}$ is not isolated, we can pick distinct points $x_{n_1 ... n_k 0}$ and $x_{n_1 ... n_k 1}$ (both $\neq x_{n_1 ... n_k}$) – in $\text{int}\, F_{n_1 ... n_k}$ and use the Lemma 3.5 to pick disjoint closed balls $F_{n_1 ... n_k 0}$ and $F_{n_1 ... n_k 1}$ (centered at $x_{n_1,,n_k 0}$ and $x_{n_1 ... n_k 1}$) and both $\subseteq$ $\text{int}\, F_{n_1 ... n_k}$. We can then shrink the balls, if necessary, so that each has diameter $< \frac{1}{2^k}$. ∙

Corollary 3.7 If (X, d) is a nonempty complete separable metric space with no isolated points, then $|X| = c$.

Proof Theorem II.5.21 (using separability) tells us that $|X| \leq c$; Theorem 3.6 gives us $|X| \geq c$. ∙

The following corollary gives us another variation on the basic result.

Corollary 3.8 If (X, d) is an uncountable complete separable metric space, then $|X| = c$. (*So we might say that "the Continuum Hypothesis holds among complete separable metric spaces."*)

Proof Since (X, d) is separable, Theorem II.5.21 gives us $|X| \leq c$.

Call a point $x \in X$ a <u>condensation point</u> if every neighborhood of x is uncountable. Let C be the set of all condensation points in X. Each point $a \in X - C$ has a countable open neighborhood O. Each point of O is also a non-condensation point, so $a \in O \subseteq X - C$. Therefore $X - C$ is open so C is closed and (C, d) is a complete metric space.

Since (X, d) is separable (and therefore second countable), $X - C$ is also second countable and therefore Lindelöf. Since $X - C$ can be covered by countable open sets, it can be covered by countably many of them, so $X - C$ is countable: therefore $C \neq \emptyset$ (in fact, C must be uncountable).

Finally, (C, d) has no isolated points: if $b \in C$ were isolated <u>in C</u>, then there would be an open set O in X with $O \cap C = \{b\}$. Since $O - \{b\} \subseteq X - C$, O would be countable – which is impossible since b is a condensation point.

Corollary 3.7 therefore applies to (C, d), and therefore $|X| \geq |C| \geq c$. ∙

Why was the idea of a "condensation point" introduced? Will the proof work if, throughout, we replace "condensation point" with "non-isolated point?" If not, precisely where would the proof break down?

The next corollary answers a question we raised earlier: is there a metric d' on $\mathbb{Q}$ which is equivalent to the usual metric d but for which $(\mathbb{Q}, d')$ is complete – that is, is $\mathbb{Q}$ "completely metrizable?"

Corollary 3.9 $\mathbb{Q}$ is not completely metrizable.

Proof Suppose d' is a metric on $\mathbb{Q}$ equivalent to the usual metric d, so that $\mathcal{T}_d = \mathcal{T}_{d'}$. Then $(\mathbb{Q}, d')$ is a nonempty metric space and, since $d \sim d'$, the space $(\mathbb{Q}, d')$ has no isolated points (no set $\{q\}$ is in $\mathcal{T}_d = \mathcal{T}_{d'}$: "isolated point" is a topological notion.). If $(\mathbb{Q}, d')$ were complete, Theorem 3.6 would imply that $\mathbb{Q}$ is uncountable. ∙

Exercises

E1. Prove that in a pseudometric space (X, d), the following are equivalent:

a) every Cauchy sequence is eventually constant
b) (X, d) is complete and $\mathcal{T}_d$ is the discrete topology
c) for every $A \subseteq X$, a Cauchy sequence in A converges to a point in A
(i.e., every subspace of (X, d) is complete).

E2. Suppose that X is a dense subspace of the pseudometric space (Y, d) and that every Cauchy sequence in X converges to some point in Y. Prove that (Y, d) is complete.

E3. Suppose that (X, d) is a metric space and that (x_n) is a Cauchy sequence with only finitely many distinct terms. Prove that (x_n) is eventually constant, i.e., that for some $n \in \mathbb{N}$, $x_n = x_{n+1} = \ldots$

E4. Let p be a fixed prime number. On the set $\mathbb{Q}$ of rationals, the p-adic absolute value $|\ |_p$ (sometimes called the p-adic norm) is defined by:

If $0 \neq x \in \mathbb{Q}$, we write $x = \frac{p^k m}{n}$ for integers k,m,n, where p does not divide m or n, and let and define $|x|_p = p^{-k} = \frac{1}{p^k}$. (*Of course, k may be negative.*). We define $|0|_p = 0$.

For all $x, y \in \mathbb{Q}$,

a) $|x|_p \geq 0$ and $|x|_p = 0$ iff $x = 0$
b) $|xy|_p = |x|_p \cdot |y|_p$
c) $|x + y|_p \leq |x|_p + |y|_p$
d) $|x + y|_p \leq \max\{|x|_p, |y|_p\} \leq x|_p + |y|_p$

The p-adic metric d_p is defined on $\mathbb{Q}$ by $d_p(x, y) = |x - y|_p$

Prove or disprove that $(\mathbb{Q}, d_p)$ is complete.

E5. Suppose that (X, d) is complete, that $(Y, \mathcal{T})$ is a Hausdorff space and that $f : X \to Y$ is continuous. Suppose that $F_1 \supseteq F_2 \supseteq \ldots \supseteq F_n \supseteq \ldots$ is a sequence of closed sets for which $\text{diam}(F_n) \to 0$. Prove that $\bigcap_{n=1}^{\infty} f[F_n] = f[\bigcap_{n=1}^{\infty} F_n]$.

E6. A metric space (X, d) is called locally complete if every point x has a neighborhood N_x (necessarily closed) which is complete. Prove that if (X, d) is a locally complete dense subspace the complete metric space (Y, d), then X is open in Y.
(*Hint: it may be helpful to recall that in any space* $(X, \mathcal{T})$*: if O is open and D is dense, then* *cl* $(O \cap D) =$ *cl(O)*.)

4. The Contraction Mapping Theorem

Definition 4.1 A point $x \in X$ is called a fixed point for the function $f : X \to X$ if $f(x) = x$. We say that a topological space X has the fixed point property if every continuous $f : X \to X$ has a fixed point.

It is easy to check that the fixed point property is a topological property.

Example 4.2

1) A function $f : \mathbb{R} \to \mathbb{R}$ has a fixed point if and only if the graph of f intersects the line $y = x$.

It's easy to see that if χ_A is the characteristic function of a set $A \subseteq \mathbb{R}$, then A is the set of fixed points for the function $f(x) = x \cdot \chi_A(x)$. *Is every set $A \subseteq \mathbb{R}$ the set of fixed points for some continuous function $f : \mathbb{R} \to \mathbb{R}$? Are there any restrictions on the cardinality of A?*

2) The interval $[a, b]$ has the fixed point property. To see this, suppose that $f : [a, b] \to [a, b]$ is continuous. If $f(a) = a$ or $f(b) = b$, then f has a fixed point – so without loss of generality, we assume that $f(a) > a$ and $f(b) < b$ and define $g(x) = f(x) - x$. Since g is continuous and $g(a) = f(a) - a > 0$ and $g(b) = f(b) - b < 0$, the Intermediate Value Theorem (from calculus) tells us that there is a point $c \in (a, b)$ with $g(c) = f(c) - c = 0$, that is, $f(c) = c$. So f has a fixed point.

3) The Brouwer Fixed Point Theorem generalizes the preceding example. It states that the solid ball $D^n = \{x \in \mathbb{R}^n : d(x, 0) \leq 1\}$ has the fixed point property. (In the case $n = 1$, D^n is homeomorphic to $[a, b]$.) The proof of Brouwer's theorem is much more difficult and the usual proofs use technique from algebraic topology.

In fact, the theorem can be generalized even further as the Schauder Fixed Point Theorem: every nonempty compact convex subset K of a Banach space has the fixed point property.

We will see that knowing whether a function $f : X \to X$ has a fixed point can sometimes be very useful. So we will look at an important kind of mapping for which this is true.

Definition 4.3 A function $f : (X, d) \to (X, d)$ is a contraction mapping (or just a contraction, for short) if there is a constant α with $0 < \alpha < 1$ and such that for all $x, y \in X$, $d(f(x), f(y)) \leq \alpha d(x, y)$.

Notice that a contraction f is automatically continuous: for every $\epsilon > 0$, we can choose $\delta = \epsilon$. Then, for any $x \in X$, $d(x, y) < \delta$ implies $d(f(x), f(y)) < \alpha\delta < \epsilon$.

In fact, the choice of δ depends only on ϵ and not the point x. In this case we say that f is uniformly continuous. The definition of the continuity of f reads:

$$\forall x \ \forall \epsilon \ \exists \delta \ \forall y \ (d(x, y) < \delta \Rightarrow d(f(x), f(y)) < \epsilon)$$

The definition of the uniform continuity of f demands more:

$$\forall\epsilon\ \exists\delta\ \forall x\ \forall y\ (d(x,y) < \delta \Rightarrow d(f(x), f(y)) < \epsilon)$$

Example 4.4

1) $f : [0, \frac{1}{4}] \to [0, \frac{1}{4}]$ given by $f(x) = x^2$ is a contraction since, for any $x, y \in [0, \frac{1}{4}]$, $|f(x) - f(y)| = |x^2 - y^2| = |x + y||x - y| \leq \frac{1}{2}|x - y|$.

2) However, $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ is not a contraction. To be a contraction would require that there exist some $\alpha \in (0, 1)$ such that the inequality

$|x^2 - y^2| \leq \alpha|x - y|$ holds for all choices of x, y, which would mean that

$|x + y| \leq \alpha$ would have to hold for all choices of $x \neq y$ (which is false).

One can also see that f is not a contraction by noting that $f : \mathbb{R} \to \mathbb{R}$ is not uniformly continuous. For example, if $\epsilon = 2$, then no suitable δ can be found. In fact, for any choice of $\delta > 0$, we can let $x = \frac{2}{\delta}$ and $y = \frac{2}{\delta} + \frac{\delta}{2}$. Then $|f(x) - f(y)| = 2 + \frac{\delta^2}{4} > 2$.

Theorem 4.5 (The Contraction Mapping Theorem) If (X, d) is a nonempty complete metric space and $f : X \to X$ is a contraction, then f has a unique fixed point.

The Contraction Mapping Theorem is often a useful tool to produce a fixed point. Moreover, the proof of the Theorem 4.5 is also important since it indicates how to make some useful numerical estimates (*See Picard's Theorem, below, and the example which follows it.*)

Proof Suppose $0 < \alpha < 1$ and that $d(f(x), f(y) \leq \alpha d(x, y)$ for all $x, y \in X$. Pick any point $x_0 \in X$ and repeatedly apply the function f, defining

$$\begin{aligned}
x_1 &= f(x_0), \\
x_2 &= f(x_1) = f(f(x_0)) = f^2(x_0) \\
&\cdots \\
x_n &= f(x_{n-1}) = \ldots = f^n(x_0) \\
&\cdots
\end{aligned}$$

We claim that the sequence (x_n) is Cauchy and converges to a fixed point for f. (Intuitively, you can think of a "control system" where the initial input is x_0 and each output becomes the new input in a "feedback loop." The system, we argue, must approach a "steady state" where "input = output.")

Suppose $\epsilon > 0$. We estimate the size of $d(x_n, x_m)$. (Without loss of generality (*why?*), we can assume $m > n$.

$$\begin{aligned}
d(x_n, x_m) &= d(f^n(x_0), f^m(x_0)) = d(f(f^{n-1}(x_0)), f(f^{m-1}(x_0))) \\
&\leq \alpha\, d(f^{n-1}(x_0), f^{m-1}(x_0)) \leq \alpha^2\, d(f^{n-2}(x_0), f^{m-2}(x_0)) \\
&\leq\ \ldots\ \leq\ \alpha^n d(x_0, f^{m-n}(x_0)) = \alpha^n\, d(x_0, x_{m-n})
\end{aligned}$$

$$\leq \alpha^n \left(d(x_0,x_1) + d(x_1,x_2) + ... + d(x_{m-n-1}, x_{m-n})\right)$$

$$\leq \alpha^n \left(d(x_0,x_1) + \alpha d(x_0,x_1) + \alpha^2 d(x_0,x_1) + ... + \alpha^{m-n-1} d(x_0,x_1)\right)$$

$$= \alpha^n\, d(x_0,x_1)\,(1 + \alpha + \alpha^2 + ... + \alpha^{m-n-1}) < \alpha^n d(x_0,x_1) \textstyle\sum_{k=0}^{\infty} \alpha^k$$

$$= \tfrac{\alpha^n d(x_0,x_1)}{1-\alpha} \to 0 \text{ as } n \to \infty.$$

Therefore if $m > n >$ some N, we have $d(x_n, x_m) < \epsilon$, so (x_n) is Cauchy. Since (X, d) is complete, there is some $x \in X$ such that $(x_n) \to x$.

By continuity, $(f(x_n)) \to f(x)$. But $(f(x_n)) = (x_{n+1}) \to x$. Since d is a metric, the sequence $(f(x_n))$ has at most one limit, so $f(x) = x$. (*If* d *were merely a pseudometric, we could only conclude that* $d(x, f(x)) = 0$.)

If y is a fixed point for f, then $d(x,y) = d(f(x), f(y)) \leq \alpha\, d(x,y)$. This implies that $d(x,y) = 0$ (since $\alpha < 1$) so $x = y$ – that is, the fixed point x is unique. (*If* d *were merely a pseudometric,* f *could have several fixed points at distance* 0 *from each other.)* ●

Notes about the proof

1) The proof gave us that if $m > n$, then $d(x_n, x_m) < \frac{\alpha^n d(x_0,x_1)}{1-\alpha}$ for each $n \in \mathbb{N}$. If we fix n and let $m \to \infty$, we get that $d(x_n, x) \leq \frac{\alpha^n d(x_0,x_1)}{1-\alpha}$. Thus we have a computable bound on how the how well x_n approximates the fixed point x.

2) We can think of x_n as an "approximate fixed point" in the sense that, for large n, f doesn't move the point x_n very far. To be specific,

$$\begin{aligned} d(x_n, f(x_n)) \quad &\leq d(x_n, x) + d(x, f(x_n)) = d(x_n, x) + d(f(x), f(x_n)) \\ &\leq d(x_n, x) + \alpha\, d(x, x_n) = (1+\alpha)\, d(x_n, x) \\ &\leq (1+\alpha) \tfrac{\alpha^n d(x_0,x_1)}{1-\alpha} \to 0 \text{ as } n \to \infty. \end{aligned}$$

Example 4.6 Consider the following functions mapping the complete space $\mathbb{R}$ into itself:

1) $f(x) = 2x + 1$: f is (uniformly) continuous, is not a contraction, and has $x = -1$ as s unique fixed point.
2) $g(x) = x^2 + 1$ is not a contraction and has no fixed point.
3) $h(x) = x + \sin x$ has infinitely many fixed points.

We are going to use the Contraction Mapping Theorem to prove a basic theorem about the existence and uniqueness of a solution for a certain kind of problem involving differential equations. To do that, we need to work inside of the "correct" complete metric space. The following example and theorem give us that space.

Example 4.7 Let X be a topological space and $C^*(X)$ be the set of bounded continuous real-valued functions $f\colon X \to \mathbb{R}$, with the metric $\rho(f,g) = \sup\{\, |\, f(x) - g(x)\, | : x \in X\}$. (*Since*

f, g are bounded, this sup is always a real number. It is easy to check that ρ is a metric.) The metric ρ is called the metric of "uniform convergence" since $\rho(f,g) < \epsilon$ implies that $|f(x) - g(x)| < \epsilon$ at every point $x \in X$ – that is, f is "uniformly close" to g.

In the specific case of $(C^*(\mathbb{R}), \rho)$, the statement that a sequence $(f_n) \overset{\rho}{\to} f$ simply states – in the familiar language of advanced calculus – that (f_n) converges to f uniformly on $\mathbb{R}$.

Theorem 4.8 $(C^*(X), \rho)$ is complete.

Proof Suppose that (f_n) is a Cauchy sequence in $(C^*(X), \rho)$: given $\epsilon > 0$, there is some N so that $\rho(f_n, f_m) < \epsilon$ whenever $n, m > N$. In particular, this means that for any fixed $x \in X$, $\rho(f_n(x), f_m(x)) < \epsilon$ whenever $n, m > N$. Therefore $(f_n(x))$ is a Cauchy sequence in $\mathbb{R}$, so $(f_n(x)) \to$ some $r_x \in \mathbb{R}$. We define $f(x) = r_x$. In this way we get a function $f : \mathbb{R} \to \mathbb{R}$. To complete the proof, we claim that $f \in C^*(X)$ and that $(f_n) \overset{\rho}{\to} f$

First we argue that f is bounded. Pick N so that $\rho(f_n, f_m) < 1$ whenever $n, m \geq N$. For $m = N$, this gives $\rho(f_n, f_N) < 1$ for $n \geq N$ and therefore, for every $x \in X$, $|f_n(x) - f_N(x)| < 1$. If we now let $n \to \infty$, we conclude that $|f(x) - f_N(x)| \leq 1$ for every x. Since f_N is bounded, there is a constant M such that $|f_N(x)| \leq M$ for every x. We therefore have, for every x :

$$-1 \leq f(x) - f_N(x) \leq 1$$

$$-M - 1 \leq f_N(x) - 1 \leq f(x) \leq f_N(x) + 1 \leq M + 1$$

$$|f_N(x)| \leq M + 1$$

So f is bounded.

We now claim that $(f_n) \overset{\rho}{\to} f$. (*Note: Technically, this is an abuse of notation – because ρ has only been defined on $C^*(X)$ and we don't yet know that f is in $C^*(X)$! However, the same definition of ρ would make sense on any collection of* <u>*bounded*</u> *real-valued functions, continuous or not. We are using that observation here.*) Let $\epsilon > 0$ and pick N so that $\rho(f_n, f_m) < \frac{\epsilon}{2}$ whenever $m, n \geq N$. Then $|f_n(x) - f_m(x)| < \frac{\epsilon}{2}$ whenever $n, m \geq N$. Letting $m \to \infty$, we get that

$$|f_n(x) - f(x)| \leq \tfrac{\epsilon}{2} \text{ for all } x \text{ and all } n \geq N.$$

Therefore $\rho(f_n, f) < \epsilon$ if $n \geq N$.

Finally, we claim that f is continuous. Suppose $a \in X$ and $\epsilon > 0$. Pick N so that if $n \geq N$ then $\rho(f_n, f) < \frac{\epsilon}{3}$. This implies that when $n \geq N$, $|f_n(x) - f(x)| < \frac{\epsilon}{3}$ for every $x \in X$. Since the function f_N is continuous at a, we can find a neighborhood W of a in X such that $|f_N(x) - f_N(a)| < \frac{\epsilon}{3}$ if $x \in W$.

Therefore, if $x \in W$, we have

$$|f(x) - f(a)| \leq |f(a) - f_N(a)| + |f_N(a) - f_N(x)| + |f_N(x) - f(x)| < \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3} = \epsilon$$

so f is continuous at a. •

Theorem 4.9 (Picard's Theorem) Suppose $f : D \to \mathbb{R}$ is continuous, where D is a closed box in $\mathbb{R}^2$ and let $(x_0, y_0) \in \operatorname{int} D$. Suppose that there exists a constant M such that for all (x_1, y_1) and (x_1, y_2) in D

$$| \, f(x_1, y_1) - f(x_1, y_2) \, | \ \le M \, | \, y_1 - y_2 \, | \qquad \text{(L)}$$

Then

i) there exists an interval $I = [x_0 - a, x_0 + a] \ = \ \{x\colon | \, x - x_0 \, | \ \le a\}$, and
ii) there exists a unique differentiable function $g : I \to \mathbb{R}$

such that $\begin{cases} y_0 = g(x_0) \\ g'(x) = f(x, g(x) \end{cases}$ for $x \in I$

In other words, on some interval I centered at x_0, there is a unique solution to the initial value problem

$$\begin{cases} y' = f(x, y) \\ y(x_0) = y_0 \end{cases} \qquad (*)$$

Before beginning the proof, we want to make some comments about the hypotheses.

1) The condition (L) on the function f is called Lipschitz condition in the variable y. It may seem a bit obscure. For our purposes, it's enough to note that (L) holds if the partial derivative f_y exists and is continuous on the box D. In that case, we know (from advanced calculus) that $|f_y| \le$ some constant M on D. We can think of $f(x_1, y)$ as a single-variable function of y and use the ordinary Mean Value Theorem to get that, for some z between y_1 and y_2,

$$|f(x_1, y_1) - f(x_1, y_2)| = |f_y(x_1, z)| \cdot |y_1 - y_2| \ \ \le M|y_1 - y_2|$$

2) As a concrete example, suppose $f(x, y) = y - x$, that $(x_0, y_0) = (0, 0)$, and that (say) $D = [-1, 1] \times [-1, 1]$. Since $f_y = 1$, the preceding comment 1) assures us that condition (L) holds.

Picard's Theorem asserts that there is a unique differentiable function $y = g(x)$, defined on some interval $[-a, a]$ for which $g(0) = 0$ and for which $g'(x) = f(x, g(x))$.

In other words, the "initial value problem"

$$\begin{cases} y' = y - x \\ y(0) = 0 \end{cases}$$

has a unique solution defined on some interval containing the origin. We shall see, moreover, that the proofs of Picard's Theorem and the Contraction Mapping Theorem can actually help us to find such a solution. Once a solution is actually found, it might turn out to be valid on some interval larger than the interval I that comes out of the proof.

Proof of Picard's Theorem

We begin by picking some things.

The constant M is given in the hypotheses.

Pick K so that $f(x, y) \leq K$ on the set D. (*We will prove later that a continuous function on a closed box in* $\mathbb{R}^2$ *must be bounded. For now, we assume that as a fact taken from advanced calculus.*)

Pick a constant $a > 0$ so that

i) $aM < 1$ and
ii) $\{x: x - x_0| \leq a\} \times \{y: y - y_0 \leq Ka\} \subseteq D$

Let $I = \{x: |x - x_0| \leq a\} = [x_0 - a, x_0 + a]$

Next, we transform the initial value problem $(*)$ into an equation involving an integral equation $(**)$ rather than a differential equation.

Suppose $y = g(x)$ is a continuous function on I and $g(x)$ satisfies (*). Since f is continuous, we know that $g'(x) = f(x, g(x))$ is continuous. (*Why?*) Therefore the Fundamental Theorem of Calculus gives us that for $x \in I$, $g(x) - g(x_0) = \int_{x_0}^{x} f(t, g(t))\, dt$, so

$$g(x) = y_0 + \int_{x_0}^{x} f(t, g(t))\, dt \quad (**)$$

On the other hand, if $y = g(x)$ is a continuous function on I and $g(x)$ satisfies the equation (**), then $y_0 = g(x_0)$ and, since $f(t, g(t))$ is continuous, the Fundamental Theorem gives that for $x \in I$, $g'(x) = f(x, g(x))$ so the function $g(x)$ satisfies (*).

It is equivalent, then, for us to find a g satisfying (**) rather than (*). (*Note that the initial condition* $y_0 = g(x_0)$ *is "built into" the single equation* $(**)$.)

We consider $(C^*(I), \rho)$, where ρ is the metric of uniform convergence. By Theorem 4.8, this space is complete. Let $B = \{g \in C^*(I) : \forall t \in I,\ |\, g(t) - y_0 \,| \leq Ka\}$. (*We want to consider only the subset* B *because if* $g \in B$ *and* $t \in I$*, then we will have* $(t, g(t)) \in D$ *and* $f(t, g(t))$ *will be defined.*) Notice that $B \neq \emptyset$ since the constant function $g(x) \equiv y_0$ is certainly in B.

We claim (B, ρ) is also complete. To see that, we prove that B is closed in $C^*(X)$. If $h \in \text{cl}\, B$, then there is sequence of functions (g_n) in B such that $(g_n) \to h$ in the metric ρ. Therefore for all $x \in I$, $g_n(x) \to h(x)$, so (subtracting y_0) we get

$$|g_n(x) - y_0\,| \;\to\; |\,h(x) - y_0\,|$$

Since $|\,g_n(x) - y_0\,| \le Ka$ for each $x \in I$, then $|\,h(x) - y_0\,| \le Ka$ for each $x \in I$. Therefore $h \in B$, so B is closed.

For $g \in B$, define a function $h = T(g)$ by

$$h(x) = T(g)(x) = y_0 + \int_{x_0}^{x} f(t, g(t))\, dt.$$

Notice that h is continuous (in fact, differentiable) and also that, for every $x \in I$,

$$|h(x)| \le |y_0| + |\int_{x_0}^{x} f(t, g(t))\, dt\,| \le |y_0| + \int_{x_0}^{x} |f(t, g(t))|\, dt \le |y_0| + \int_{x_0}^{x} K\, dt \le |y_0| + Ka,$$

so h is bounded. Therefore $T : B \to C^*(I)$. In fact, more is true: $T\colon B \to B$, because

$$\forall x \in I,\; |T(g)(x) - y_0| = |\,h(x) - y_0| = |\int_{x_0}^{x} f(t, g(t))\, dt\,| \le Ka$$

We now claim that $T : (B, \rho) \to (B, \rho)$ is in fact a contraction. To see this, we compute distances: if g_1 and $g_2 \in B$, then

$$\begin{aligned}
\rho(T(g_1), T(g_1)| &= \sup\{\,|\,T(g_1(x)) - T(g_2(x))|\; : x \in I\} \\
&= \sup\{\,|\int_{x_0}^{x} f(t, g_1(t)) - f(t, g_2(t))\, dt|\; : x \in I\} \\
&\le \sup\{\,|\int_{x_0}^{x} |f(t, g_1(t)) - f(t, g_2(t))|\; dt| : x \in I\,\} \\
&\le \sup\{\,|\int_{x_0}^{x} M\, |g_1(t) - g_2(t)|\, dt\,| : x \in I\} \quad (\text{from (L)}) \\
&\le \sup\{\,|\int_{x_0}^{x} M\rho(g_1, g_2)\, dt\,| : x \in I\} \\
&= \sup\{\,M\rho(g_1, g_2)\;|\,x - x_0| : x \in I\} \\
&= aM\rho(g_1, g_2) \\
&= \alpha\rho(g_1, g_2),\; \text{where } \alpha = aM \;<\; 1 \;\text{(by choice of } a).
\end{aligned}$$

Since B is a nonempty, complete metric space, the Contraction Mapping Theorem tells us that there is a unique function $g \in B$ such that $T(g) = g$. By definition of T, that simply means:

$$\forall x \in I,\; g(x) = T(g)(x) = y_0 + \int_{x_0}^{x} f(t, g(t))dt$$

which is precisely condition (**). •

Example 4.10 The proof of Picard's Theorem and the numerical estimates from the proof of the Contraction Mapping Theorem can both be used to get useful information about a concrete initial problem. To illustrate, we will consider

$$\begin{cases} y' = y - x \\ y(0) = 0 \end{cases}$$

and find a solution valid on some interval containing 0.

We begin by choosing a box D with $(0,0)$ in its interior. Rather arbitrarily, we select $D = [-1,1] \times [-1,1]$. Since $|f(x,y)| = |y - x| \leq |x| + |y| \leq 2$ on D, we can use $K = 2$ in the proof.

Because $|f_y(x,y)| = 1$ throughout D, the Lipschitz condition (L) will be satisfied with $M = 1$.

Following the proof of Picard's Theorem, we now choose a constant a so that

i) $aM < 1$ and
ii) $\{x : |x| \leq a\} \times \{y : |y| \leq aK = 2a\} \subseteq D$.

Somewhat arbitrarily, we choose $a = \frac{1}{2}$, so that in the proof $I = [-\frac{1}{2}, \frac{1}{2}]$. We then have $B = \{g \in C^*(I) : |g(x) - 0| \leq Ka\} = \{g \in C^*(I) : |g(x)| \leq 1 \text{ for all } x \in I\}$.

Finally, we then choose any function $g_0 \in B$; in the interests of simplicity, we might as well choose g_0 to be the constant function 0 on I.

According to the proof of the Contraction Mapping Theorem, the sequence of functions $(g_n) = (T^n(g_0)) \xrightarrow{\rho} g$, where g is a fixed point for T, which is the solution to our initial value problem. We can calculate:

$$g_1(x) = y_0 + \int_{x_0}^{x} f(t, g_0(t))\,dt = 0 + \int_0^x f(t,0)\,dt = \int_0^x -t\,dt = -\frac{x^2}{2}$$

$$g_2(x) = T(g_1(x)) = 0 + \int_0^x f(t, -\tfrac{t^2}{2})\,dt = \int_0^x -\frac{t^2}{2} - t\,dt = -\frac{x^3}{6} - \frac{x^2}{2}$$

$$g_3(x) = T(g_2(x)) = \int_0^x f(t, -\tfrac{t^3}{6} - \tfrac{t^2}{2})\,dt = \int_0^x -\frac{t^3}{6} - \frac{t^2}{2} - t\,dt = -\frac{x^4}{24} - \frac{x^3}{6} - \frac{x^2}{2}$$

and, in general,

$$g_n(x) = T(g_{n-1}(x)) = \ldots = -\frac{x^2}{2!} - \frac{x^3}{3!} - \ldots - \frac{x^{n+1}}{(n+1)!}$$

and the functions $g_n(x)$ converge (uniformly) to the solution we seek.

In this particular problem, we are lucky enough to recognize that the functions $g_n(x)$ are just the partial sums of the series $\sum_{n=2}^{\infty} -\frac{x^n}{n!} = 1 + x - \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x - e^x$. Therefore $g(x) = 1 + x - e^x$ is a solution of our initial value problem, and we know it is valid on the interval $I = [-\frac{1}{2}, \frac{1}{2}]$. (*One may check by direct substitution that the solution is correct – and that it turns out to be valid over the whole interval* $\mathbb{R}$.)

Even if we couldn't recognize a neat formula for the limit of the $g_n(x)$'s, we could make some definite approximations. From the proof of the Contraction Mapping Theorem, we know that $\rho(g_n, g) \leq \frac{\alpha^n \rho(g_0, g_1)}{1-\alpha}$. In this example, we have $\alpha = aM = \frac{1}{2}$, so that
$\rho(g_n, g) \leq \frac{\frac{1}{2^n}\rho(g_0,g_1)}{1-\frac{1}{2}} = \frac{1}{2^{n-1}} \sup\{|0 - (-\frac{x^2}{2})| : x \in I\} = \frac{1}{2^{n-1}} \cdot \frac{1}{2^3} = \frac{1}{2^{n+2}}$.
Therefore $g_n(x)$ is uniformly within distance $\frac{1}{2^{n+2}}$ of $g(x) = 1 + x - e^x$ on the interval $[-\frac{1}{2}, \frac{1}{2}]$.

Finally, notice that our initial choice of $g_0 \in B$ was an arbitrary one. Since $|\sin x| \leq 1$, the function $\sin \in B$ and we could just as well have chosen $g_0(x) = \sin x$. The iterative process would necessarily lead us to the same solution g, since the fixed point of T is unique; but of course, the resulting series representation for g would look quite different $-$ for example, it would not be a power series. •

The Contraction Mapping Theorem can be used to prove other results $-$ for example, the Implicit Function Theorem. (See *Topology,* James Dugundji, for the details.)

Exercises

E7. a) Suppose $f : [a,b] \to [a,b]$ is differentiable and suppose that there is a constant $K < 1$ such that $|f'(x)| \leq K$ for all $x \in [a,b]$. Prove that f is a contraction (and therefore has a unique fixed point.)

b) Give an example of a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that
$$|f(x) - f(y)| < |x - y| \text{ for}$$
all $x \neq y \in \mathbb{R}$ but such that f has no fixed point.

(*Note: b) implies that f is not a contraction mapping. The contraction mapping theorem would fail if we allowed $\alpha = 1$ in the definition of contraction mapping.*)

E8. Let $f: (X,d) \to (X,d)$, where (X,d) is a nonempty complete metric space. Let f^k denote the "k^{th} iterate of f" – that is, f composed with itself k times.

a) Suppose that $\exists k \in \mathbb{N}$ for which f^k is a contraction. Then, by the Contraction Mapping Theorem, f^k has a unique fixed point p. Prove that p is also the unique fixed point for f.

b) Prove that the function $\cos: \mathbb{R} \to \mathbb{R}$ is not a contraction.

c) Prove that $\cos^k$ is a contraction for some $k \in \mathbb{N}$.

(*Hint: the Mean Value Theorem may be helpful.*)

d) Let $k \in \mathbb{N}$ be such that $g = \cos^k$ is a contraction and let p be the unique fixed point of g. By a), p is also the unique solution of the equation $\cos x = x$. Start with 0 as a "first approximation" for p and use the technique in the proof of the Contraction Mapping Theorem to find an $n \in \mathbb{N}$ so that $|g^n(0) - p| < 0.00001$.

e) For this n, use a calculator to evaluate $g^n(0)$. (*This "solves" the equation $\cos x = x$ with $|Error| < 0.00001$.*)

E9. Consider the differential equation $y' = x + y$ with the initial condition $y(0) = 1$. Choose a suitable rectangle *D* and suitable constants *K*, *M*, and *a* as in the proof of Picard's Theorem. Use the technique in the proof of the contraction mapping theorem to find a solution for the initial value problem.