Example 1 is almost identical to the "closed exchange economy" example handed out in class during the first week of the course – although there are a few more comments and updates in light of what we have learned since then.

Then, Example 2 illustrates (on a very small scale) some of the ideas behind making a more complicated model of an "open" economy.

Example 1 **Closed Exchange Economy**

Suppose an economy has only 4 sectors: agriculture (A), energy (E), manufacturing (M) and transportation (T).

In this simple "closed exchange economy," we assume that all goods are manufactured and used among these four sectors.

The production/consumption among the sectors is summarized in an exchange matrix

$$\begin{array}{c} \text{Production of sector} \\ \begin{array}{cccc} \downarrow & \downarrow & \downarrow & \downarrow \\ A & E & M & T \end{array} \end{array}$$

$$\begin{bmatrix} .65 & .30 & .30 & .20 \\ .10 & .10 & .15 & .10 \\ .25 & .35 & .15 & .30 \\ 0 & .25 & .40 & .40 \end{bmatrix} \quad \to \text{consumed by} \quad \begin{cases} A \\ E \\ M \\ T \end{cases}$$

Going down a column, we can read off what part of a given sector's production goes to each of the other sectors: for example, the production of the manufacturing sector (M) is consumed as

30% to the agriculture sector,
15% to the energy sector,
15% to the manufacturing sector (it uses up some of its own goods), and
40% to the transportation sector.

Therefore, of course, the sum of each column is 1 (100%).

Going across a particular row, we can read off what part of each sector's production is consumed by one particular sector: for example, the energy sector consumes 10% of the production of the agriculture, energy and transportation sectors and 15% of the production of the manufacturing sector. Each row is a consumption vector for one of the sectors.

Suppose p_A, p_E, p_M, p_T represent the total production of goods in each sector (measured for convenience, say, in \$). Is it possible to set production levels for the four sectors so that "everybody's happy" $-$ that is, every sector gets what it needs (using the value of its own goods to pay) and nothing is left over? If so, then the values p_A, p_E, p_M, p_T give an <u>equilibrium</u> for this "closed exchange economy."

The cost to sector A for what it needs is

$$.65P_A + .30P_E + .30P_M + .20P_T$$

This expense must be paid for by the value of sector A's goods, P_A, so

$$\begin{array}{ll}
 & .65P_A + .30P_E + .30P_M + .20P_T = P_A, \quad \text{and similarly} \\
(*) & .10P_A + .10P_E + .15P_M + .10P_T = P_E \\
 & .25P_A + .35P_E + .15P_M + .30P_T = P_M \\
 & \qquad\quad .25P_E + .40P_M + .40P_T = P_T
\end{array}$$

Rearranging these equations gives the linear system

$$\begin{array}{ll}
 & -.35P_A + .30P_E + .30P_M + .20P_T = 0 \\
(**) & .10P_A - .90P_E + .15P_M + .10P_T = 0 \\
 & .25P_A + .35P_E - .85P_M + .30P_T = 0 \\
 & \qquad\quad .25P_E + .40P_M - .60P_T = 0
\end{array}$$

whose augmented matrix is

$$\begin{bmatrix} -.35 & .30 & .30 & .20 & 0 \\ .10 & -.90 & .15 & .10 & 0 \\ .25 & .35 & -.85 & .30 & 0 \\ 0 & .25 & .40 & -.60 & 0 \end{bmatrix}.$$ The row reduced echelon form (with entries <u>rounded</u> to 2 decimal places for convenience) is

$$\begin{bmatrix} 1 & 0 & 0 & -2.03 & 0 \\ 0 & 1 & 0 & -0.53 & 0 \\ 0 & 0 & 1 & -1.17 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Writing $\approx$ instead of $=$ (since I rounded decimals), we have

$P_A \approx 2.03P_T$
$P_E \approx 0.53P_T$
$P_M \approx 1.17P_T$
P_T is free

Therefore: If a total production P_T ($) for the transportation sector is determined, then the productions of the other sectors can be adjusted to create an equilibrium. Of course, one would choose a value of $P_T \geq 0$ for the solutions to be economically feasible.

Some additional comments about Example 1

The system of equations (*) in Example 1 can be written in matrix form:

$$E\boldsymbol{p} = \boldsymbol{p}$$

where $\boldsymbol{E}$ is the "exchange matrix" $\begin{bmatrix} .65 & .30 & .30 & .20 \\ .10 & .10 & .15 & .10 \\ .25 & .35 & .15 & .30 \\ 0 & .25 & .40 & .40 \end{bmatrix}$ and $\boldsymbol{p} = \begin{bmatrix} p_A \\ p_E \\ p_M \\ p_T \end{bmatrix}$.

Each row is a consumption vector for one of the sectors. For example, the row $[.10 \ .10 \ .15 \ .10]$ indicates that the energy sector consumes 10% of the production of the agriculture and energy sectors, 15% of the production of the manufacturing sector, and 10% of the production of the transportation sector.

We could rewrite the matrix equation as follows:

$$E\boldsymbol{p} = I\boldsymbol{p} \quad \text{(where } I \text{ is the identity matrix } I_4\text{)}$$
$$E\boldsymbol{p} - I\boldsymbol{p} = \mathbf{0}$$
$$(E - I)\boldsymbol{p} = \mathbf{0}$$

(or equivalently, $(I - E)\boldsymbol{p} = \mathbf{0}$ if we multiply both sides of the equation by -1)

The last equation is just the matrix form of the system (**) in Example 1, because

$$E - I = \begin{bmatrix} .65 & .30 & .30 & .20 \\ .10 & .10 & .15 & .10 \\ .25 & .35 & .15 & .30 \\ 0 & .25 & .40 & .40 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -.35 & .30 & .30 & .20 \\ .10 & -.90 & .15 & .10 \\ .25 & .35 & -.85 & .30 \\ 0 & .25 & .40 & -.60 \end{bmatrix}$$

We could row reduce the augmented matrix $\begin{bmatrix} -.35 & .30 & .30 & .20 & 0 \\ .10 & -.90 & .15 & .10 & 0 \\ .25 & .35 & -.85 & .30 & 0 \\ 0 & .25 & .40 & -.60 & 0 \end{bmatrix}$ to

find the general solution. It will turn out that there are nontrivial solutions, as the following more general discussion indicates.

Here is the general version of this “closed exchange economy” model:

We have an economy divided into n sectors $S_1,\ S_2, ..., S_n$. The total production of each sector is consumed by (“exchanged with”) the other sectors. Each row of the exchange matrix is a consumption vector for one of the sectors: the entries e_{i1}, e_{i2},..., e_{ij},... e_{in} in row i give the proportion of the outputs of $S_1,\ S_2, ..., S_n$ consumed by S_i.

Fraction of Production of

$$E = \begin{array}{c} \begin{matrix} \boldsymbol{S_1} \cdots & \boldsymbol{S_j} & \cdots & \boldsymbol{S_n} \\ & \downarrow & & \end{matrix} \\ \begin{bmatrix} e_{11} \cdots & e_{1j} & \cdots & e_{1n} \\ & \vdots & & \\ e_{i1} \cdots & e_{ij} & \cdots & e_{in} \\ & \vdots & & \\ e_{n1} \cdots & e_{nj} & \cdots & e_{nn} \end{bmatrix} \end{array} \qquad \rightarrow \ \textbf{Consumed by} \begin{cases} \boldsymbol{S_1} \\ \vdots \\ \boldsymbol{S_i} \\ \vdots \\ \boldsymbol{S_n} \end{cases}$$

Of course, each entry $e_{ij} \geq 0$. Since, in this model, all the production of each sector is consumed by the other sectors, each column sum is 1: that is, for each fixed j, $\sum_{i=1}^{n} e_{ij} = 1$.

We are interested in finding a production vector $\boldsymbol{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ for which $E\boldsymbol{x} = \boldsymbol{x}$. Such an $\boldsymbol{x}$ represents an equilibrium production from each sector in which all needs are met with no surplus.

We can rewrite this equation as an equivalent homogenous system:

$$(E - I)\boldsymbol{x} = \mathbf{0}$$

Of course, we have an equilibrium if all the sectors produce nothing, that is, for the trivial solution $\boldsymbol{x} = \mathbf{0}$. So we are really interested in the nontrivial solutions (if any) for $E\boldsymbol{x} = \boldsymbol{x}$. Moreover, in the economic situation, we are interested not just in a nontrivial mathematical solution, but an “economically feasible” nontrivial solution $-$ one where all $x_i \geq 0$.

Can we always find such an $\boldsymbol{x}$? The next theorem offers a partial answer.

Theorem Let E be an $n \times n$ matrix whose entries are all nonnegative and where each column sum is 1. Then there is a nontrivial solution for $(E - I)\boldsymbol{x} = \mathbf{0}$.

Proof For each j, the j^{th} column of $(E - I)$ is the same as the j^{th} column of E except that the diagonal entry e_{jj} in column j has been replaced by $e_{jj} - 1$. Since $e_{1j} + e_{2j} + ... + e_{jj} + ... + e_{nj} = 1$, then $e_{1j} + e_{2j} + ... + (e_{jj} - 1) + ... + e_{nj} = 0$: that is, each column in $(E - I)$ has sum $= 0$.

Begin row reducing the augmented matrix $[E - I \mid \mathbf{0}]$ by adding each row to the bottom row: first add row 1, then row 2, ..., etc. This means that, in each column, the bottom entry will end up being the sum of all numbers in that column – namely, 0. So, after these steps in a row reduction, we end up with a bottom row of 0's. Therefore the matrix $E - I$ (which is $n \times n$) cannot have n pivot positions. Therefore the system homogeneous system $(E - I)\boldsymbol{x} = \mathbf{0}$ has at least one free variable, so it must have nontrivial solutions (in fact, infinitely many of them). •

Note: It is possible to prove, in addition, that there must be a nontrivial solution $\boldsymbol{x}$ where all the entries $x_i \geq 0$. But the proof is considerably harder and we can't do it here.

Examples Here are two very simple economies – only two sectors; they are here to make a mathematical point, not because they're realistic.

a) If $E = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 1 \end{bmatrix}$, then $(E - I)\boldsymbol{x} = \begin{bmatrix} -\frac{1}{2} & 0 \\ \frac{1}{2} & 0 \end{bmatrix}\boldsymbol{x} = \mathbf{0}$ has the general solution

$$\boldsymbol{x} = s\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Looking at the rows of E (the consumption vectors) we see that, unlike Sector 1, Sector 2 consumes all of its own product. Half the product of Sector 1 is "out of balance" – in the sense that Sector 1 doesn't use it and Sector 2 has nothing left with which to pay for it. An equilibrium is impossible unless Sector 1 produces nothing (*"half of nothing out of balance" is not a problem*).

And that's what the solution shows: Sector 1 must produce nothing, and Sector 2 can produce as much or little as it likes.

b) If $E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then $(E - I)\boldsymbol{x} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\boldsymbol{x} = \mathbf{0}$

Looking at the rows of E (the consumption vectors) we see that Sector 1 and Sector 2 each consume all of their own product and nothing from the other sector. The production of the sectors is "independent." Therefore, each sector can produce as much it likes and the economy will be in equilibrium.

Mathematically, every $\boldsymbol{x}$ in $\mathbb{R}^2$ is a solution: both variables in the system are free, and we can write the set of solutions as $\boldsymbol{x} = \begin{bmatrix} s \\ t \end{bmatrix}$, or $\boldsymbol{x} = s\begin{bmatrix} 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. An economically feasible solution requires choosing any values $s \geq 0$ and $t \geq 0$.

In this example the possible production vectors cannot all be described as multiples of a single vector $-$ that is, there are production vectors where one is not just a rescaling of the other.

There's a deeper theorem that can be proven in a course on stochastic processes.

Theorem Let E be an $n \times n$ matrix whose entries are all nonnegative and for which all column sums are 1. Suppose, in addition, that there is some positive integer k for which the entries in the matrix E^k are all strictly positive (that is, all are > 0, not merely ≥ 0). Then the solution set of $(E - I)\boldsymbol{x} = \mathbf{0}$ is spanned by a single vector $-$ so the general solution is $\boldsymbol{x} = t\boldsymbol{v}$ for some $\boldsymbol{v}$ in $\mathbb{R}^n$. The vector $\boldsymbol{v}$ can be chosen so that all its entries are > 0.

Therefore, if the exchange matrix satisfies the condition in the Theorem, outcomes like Examples a) and b) cannot happen.

<u>Example 2</u> Leontief Model of an Open Economy (Input/Output Model)

Assume again that an economy has n sectors $S_1, \ldots, S_n$. As before, a <u>production vector</u>

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$ gives the production of each sector (measured in $).

We can look at the general "setup" for Leontief's model of an "open economy" – a term which we will explain below. The setup has similarities to Example 1, but it also has differences: for example, we look at <u>not</u> at the total consumption of Sector j from the other sectors, but the consumption by Sector j <u>per unit</u> of Sector j product.

We can also solve the system we set up in the case of a simple example. However, proofs of the interesting theorems about "what happens in general" are beyond the scope of this course.

We begin with a <u>consumption matrix</u> C in which the j^{th} <u>column</u> lists the inputs ($) from $S_1, S_2, \ldots, S_n$ <u>consumed</u> by sector S_j <u>to produce one unit</u> ($) of output. Of course, each $c_{ij} \geq 0$.

$$\textbf{Input from} \left\{ \begin{array}{l} \boldsymbol{S_1} \;\longrightarrow \\ \vdots \\ \boldsymbol{S_i} \;\longrightarrow \\ \vdots \\ \boldsymbol{S_n} \;\longrightarrow \end{array} \right. \qquad \begin{array}{c} \begin{bmatrix} c_{11} \cdots & c_{1j} & \cdots & c_{1n} \\ & \vdots & & \\ c_{i1} \cdots & c_{ij} & \cdots & c_{in} \\ & \vdots & & \\ c_{n1} \cdots & c_{nj} & \cdots & c_{nn} \end{bmatrix} = C \\ \uparrow \qquad\quad \uparrow \qquad\quad \uparrow \\ \textbf{Consumption vectors for} \\ \boldsymbol{S_1} \cdots \boldsymbol{S_j} \cdots \boldsymbol{S_n} \end{array}$$

For example, <u>Column 1</u> $= \begin{bmatrix} c_{11} \\ \vdots \\ c_{i1} \\ \vdots \\ c_{n1} \end{bmatrix}$ lists the separate inputs ($) <u>from</u> Sectors 1 ,...,

Sector n <u>consumed by Sector 1</u> to produce one unit ($) of its own product. These "inputs" are <u>costs</u> to Sector 1.

The sum of the entries in Column 1 gives the total cost of products consumed by Sector to produce one unit of its product.

The sum of the entries in Column j represents the total input ($) from $S_1, S_2, ..., S_n$ consumed by S_j in order for S_j to produce one unit ($) of product. This is the cost to Sector j to produce one unit. If the j^{th} column sum is < 1, we say that sector S_j is profitable.

If we weight Column j with a scalar $x_j \geq 0$, then the vector $x_j \begin{bmatrix} c_{1j} \\ \vdots \\ c_{ij} \\ \vdots \\ c_{nj} \end{bmatrix} = \begin{bmatrix} x_1 c_{1j} \\ \vdots \\ x_1 c_{ij} \\ \vdots \\ x_1 c_{nj} \end{bmatrix}$ lists

the inputs ($) consumed from other sectors by sector S_j to produce x_j units ($) of its product.

If Sectors 1,2,...,n produce $x_1, x_2, ..., x_n$ units, then if we have a production vector

$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, where the $x_i \geq 0$, what does the vector $C\boldsymbol{x}$ mean?

$$C\boldsymbol{x} = x_1 \begin{bmatrix} c_{11} \\ \vdots \\ c_{i1} \\ \vdots \\ c_{n1} \end{bmatrix} + x_2 \begin{bmatrix} c_{12} \\ \vdots \\ c_{i2} \\ \vdots \\ c_{n2} \end{bmatrix} + ... + x_n \begin{bmatrix} c_{1n} \\ \vdots \\ c_{in} \\ \vdots \\ c_{nn} \end{bmatrix} = \begin{bmatrix} c_{11}x_1 + ... + c_{1n}x_n \\ \vdots \\ c_{i1}x_1 + ... + c_{in}x_n \\ \vdots \\ c_{n1}x_1 + ... + c_{nn}x_n \end{bmatrix}$$

$$= \begin{bmatrix} \textit{total (\$) consumed by Sectors } S_1, ..., S_n \ \underline{\textit{from}}\ S_1 \textit{ to produce } \boldsymbol{x} \\ \vdots \\ \textit{total (\$) consumed by Sectors } S_1, ..., S_n \ \underline{\textit{from}}\ S_i \textit{ to produce } \boldsymbol{x} \\ \vdots \\ \textit{total (\$) consumed by Sectors } S_1, ..., S_n \ \underline{\textit{from}}\ S_n \textit{ to produce } \boldsymbol{x} \end{bmatrix}$$

$C\boldsymbol{x}$ lists the inputs demanded (consumed) from each sector if they are to deliver the production vector $\boldsymbol{x}$.

If we examine the equation $C\boldsymbol{x} = \boldsymbol{x}$, then we are looking for an equilibrium value $-$ an $\boldsymbol{x}$ for which the total amount produced by each sector = the amount of its product consumed by all the other sectors. The production from each sector is not too large, not small but "just right." This is a question about a "closed" economy.

Now we add another feature to the model: an open sector. It is nonproductive; it produces nothing that sectors $S_1, \ldots, S_n$ use; the open sector simply "demands" goods produced by sectors $S_1, \ldots, S_n$.

A final demand vector $\boldsymbol{d} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix}$ tells how many units ($) the open sector demands from sectors $S, \ldots, S_n$. For example, the nonproductive sector might be the government, or perhaps a sector consisting of charitable organizations.

We want to know if it is possible to set a level of production $\boldsymbol{x}$ so that both the productive and open sectors are satisfied, with nothing left over. That is, we want to find an $\boldsymbol{x}$ so that

Leontief Open Economy Production Model

$$\underset{\substack{\uparrow \\ \text{Total production}}}{\boldsymbol{x}} = \underset{\substack{\uparrow \\ \text{Demand from} \\ \text{productive sectors} \\ \text{to produce } \boldsymbol{x}}}{C\boldsymbol{x}} + \underset{\substack{\uparrow \\ \text{Demand from} \\ \text{open sector}}}{\boldsymbol{d}} \qquad (***)$$

We can rewrite this equation in the form $(I - C)\,\boldsymbol{x} = \boldsymbol{d}$

To take a simple example, suppose the economy has 4 productive sectors $S_1, \ldots S_4$ and the consumption matrix is

$$C = \begin{bmatrix} .10 & .05 & .30 & .20 \\ .15 & .25 & .05 & .10 \\ .30 & .10 & .10 & .25 \\ .15 & .20 & .10 & .20 \end{bmatrix}$$

$\uparrow \quad \uparrow \quad \uparrow \quad \uparrow$

Consumption vectors for

$S_1 \quad S_2 \quad S_3 \quad S_4$

(<u>*Note*</u>*: The column sums in C are all < 1 – so each sector is profitable.*)

Then $$(I - C) = \begin{bmatrix} .90 & -.05 & -.30 & -.20 \\ -.15 & .75 & -.05 & -.10 \\ -.30 & -.10 & .90 & -.25 \\ -.15 & -.20 & -.10 & .80 \end{bmatrix}$$

If the demand from the open sector is $\boldsymbol{d} = \begin{bmatrix} 25000 \\ 10000 \\ 30000 \\ 50000 \end{bmatrix}$, then we want to try to solve

$$(I - C)\boldsymbol{x} = \begin{bmatrix} .90 & -.05 & -.30 & -.20 \\ -.15 & .75 & -.05 & -.10 \\ -.30 & -.10 & .90 & -.25 \\ -.15 & -.20 & -.10 & .80 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 25000 \\ 10000 \\ 30000 \\ 50000 \end{bmatrix} = \boldsymbol{d}$$

Row reducing the augmented matrix gives

$$\begin{bmatrix} .90 & -.05 & -.30 & -.20 & 25000 \\ -.15 & .75 & -.05 & -.10 & 10000 \\ -.30 & -.10 & .90 & -.25 & 30000 \\ -.15 & -.20 & -.10 & .80 & 50000 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 85580 \\ 0 & 1 & 0 & 0 & 50620 \\ 0 & 0 & 1 & 0 & 96160 \\ 0 & 0 & 0 & 1 & 103220 \end{bmatrix}$$

(*The calculations were carried out to many decimal places, but the final numbers displayed are rounded to the nearest unit* ($)).

Equivalently (but harder to compute by hand), we could try to find $(I - C)^{-1}$ and, if it exists, then compute $\boldsymbol{x} = (I - C)^{-1}\boldsymbol{d}$. (*In this example, it turns out that $I - C$ is invertible so this approach would work.*)

The equation (***) will be satisfied if S_1 produces 85580 units ($) of product, S_2 produces 50620 ($) , etc.

Here's a theorem that's too hard for us to prove:

Theorem (*using the same notation as above*) If C and $\boldsymbol{d}$ have nonnegative entries and the column sums in C are all < 1 (*that is, if every sector in profitable*), then $(I - C)$ is invertible so there is a unique production vector satisfying the equation $(I - C)\boldsymbol{x} = \boldsymbol{d}$: namely $\boldsymbol{x} = (I - C)^{-1}\boldsymbol{d}$. Moreover, this $\boldsymbol{x}$ is economically feasible in the sense that all its entries are ≥ 0.

Here is an informal, imprecise discussion that might lead you to believe the theorem is true.

1) Suppose C is any $n \times n$ matrix with entries all ≥ 0 and all column sums < 1. Then $C^m \to \mathbf{0}$ as $m \to \infty$.

$\uparrow$

(the $n \times n$ zero matrix)

This statement just means that every entry in C will be made as close to 0 as we like provided we choose a large enough value for m.

To keep things simple, we just look at the case of a 2×2 matrix $C = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Let α be the <u>largest column sum</u> in C. Our assumptions give that $0 \leq \alpha < 1$. Therefore each column sum $a + c \leq \alpha$ and $b + d \leq \alpha$.

Now an observation: suppose $\begin{bmatrix} x & y \\ z & w \end{bmatrix}$ is <u>any</u> 2×2 matrix with $x, y, z, w \geq 0$. Then $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} ax+bz & ay+bw \\ cx+dz & cy+dw \end{bmatrix}$ and this new matrix has

column sums

Column 1: $(a+c)x + (b+d)z \leq \alpha x + \alpha z = \alpha(x+z)$
Column 2: $(a+c)y + (b+d)w \leq \alpha y + \alpha w = \alpha(y+w)$

$\uparrow$

$=$ α times the old column sums of $\begin{bmatrix} x & y \\ z & w \end{bmatrix}$

If we apply this observation starting with $\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ we get that the column sums for $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = C^2$ satisfy:

sum for column 1 of $C^2 \leq \alpha$ (old column 1 sum) $= \alpha(a+c) \leq \alpha \cdot \alpha = \alpha^2$
sum for column 2 of $C^2 \leq \alpha$ (old column 2 sum) $= \alpha(b+d) \leq \alpha \cdot \alpha = \alpha^2$

.

Repeating this argument again with $\begin{bmatrix} x & y \\ z & w \end{bmatrix} = C^2$, we see that in the matrix $C^3 = C \cdot C^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot C^2$, the sum of each column is $\leq \alpha^3$.

Continuing in this way, we find that the sum of each column in C^m is $\leq \alpha^m$, and (since all the elements in C^m are nonnegative) this means that every entry in the matrix C^m is $\leq \alpha^m$. Since $0 \leq \alpha < 1$, $\alpha^m \to 0$ as $m \to \infty$. Therefore $C^m \to 0$ as $m \to \infty$.

2) We can write a matrix equation

$$I - C^{m+1} = (I - C)(I + C + C^2 + ... + C^m)$$

No hand-waving here; just multiply it out to verify:

$$\begin{aligned}
&(I - C)(I + C + C^2 + ... + C^m) \\
&\quad = (I - C)I + (I - C)C + (I - C)C^2 + ... + (I - C)C^m \\
&\quad = I^2 - CI + IC - C^2 + IC^2 - C^3 + ... + IC^m - C^{m+1} \\
&\quad = I - C + C - C^2 + C^2 ... \qquad ... - C^m + C^m - C^{m+1} \\
&\quad = I - C^{m+1}
\end{aligned}$$

3) Since $C^m \to \mathbf{0}$ as $m \to \infty$, we have that $C^{m+1} \approx \mathbf{0}$ for large m, so for large m,

$$I \approx (I - C)(I + C + C^2 + ... + C^m)$$

which means $(I + C + C^2 + ... + C^m)$ is approximately an inverse for $(I - C)$.

This suggests that $I - C$ is invertible. Moreover, $(I + C + C^2 + ... + C^m)$ has nonnegative entries and $(I - C)^{-1} \approx (I + C + C^2 + ... + C^m)$; this suggests that the elements of $(I - C)^{-1}$ are all nonnegative.

4) Of course, if $(I - C)$ is invertible and $(I - C)^{-1}$ has all nonnegative elements, then we have a unique solution $\boldsymbol{x} = (I - C)^{-1}\boldsymbol{d}$ and $(I - C)^{-1}\boldsymbol{d}$ has all nonnegative entries.

Having done all this, we can also notice that if $(I - C)^{-1} \approx (I + C + C^2 + ... + C^m)$,

then

$$\boldsymbol{x} = (I - C)^{-1}\boldsymbol{d} \approx (I + C + C^2 + ... + C^m) \cdot \boldsymbol{d} \qquad \text{or}$$

$$\boldsymbol{x} \approx \boldsymbol{d} + C\boldsymbol{d} + C^2\boldsymbol{d} + ... + C^m\boldsymbol{d}$$

(*See text, p. 154-155 for additional comments about the economic significance of* $(I - C)^{-1}$.