

Irrationality of The Square Root of Two —A Geometric Proof

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This note presents a remarkably simple proof of the irrationality of $\sqrt{2}$ that is a variation of the classical Greek geometric proof.

By the Pythagorean theorem, an isosceles right triangle of edge-length 1 has hypotenuse of length $\sqrt{2}$. If $\sqrt{2}$ is rational, some positive integer multiple of this triangle must have three sides with integer lengths, and hence there must be a *smallest* isosceles right triangle with this property. But inside any isosceles right triangle whose three sides have integer lengths we can always construct a smaller one with the same property, as shown below. Therefore $\sqrt{2}$ cannot be rational.

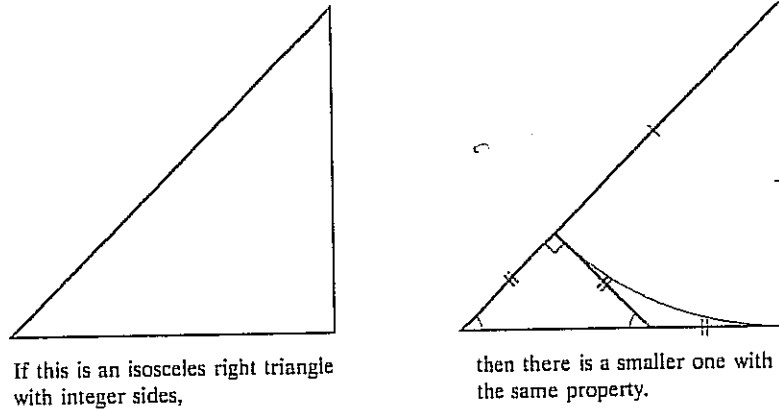


Figure 1

Construction. A circular arc with center at the uppermost vertex and radius equal to the vertical leg of the triangle intersects the hypotenuse at a point, from which a perpendicular to the hypotenuse is drawn to the horizontal leg. Each line segment in the diagram has integer length, and the three segments with double tick marks have equal lengths. (Two of them are tangents to the circle from the same point.) Therefore the smaller isosceles right triangle with hypotenuse on the horizontal base also has integer sides.

The reader can verify that similar arguments establish the irrationality of $\sqrt{n^2 + 1}$ and $\sqrt{n^2 - 1}$ for any integer $n > 1$. For $\sqrt{n^2 + 1}$ use a right triangle with legs of lengths 1 and n . For $\sqrt{n^2 - 1}$ use a right triangle with hypotenuse n and one leg of length 1.