Chapter II
Metric and Pseudometric Spaces

1. Introduction

By itself, a set doesn't have any structure. For two arbitrary sets A and B, we can ask questions like "Is $A \subseteq B$?" or "Is A equivalent to a subset of B?" but not much more. If we add additional structure to a set, it becomes more interesting. For example, if we define a "multiplication operation" $a \cdot b$ in X that satisfies certain axioms (such as $a \cdot (b \cdot c) = (a \cdot b) \cdot c$), then X becomes an algebraic structure called a group and a whole area of mathematics known as group theory begins.

We are not interested in making a set X into an algebraic system. Rather, we want additional structure on a set X so that we can talk about "nearness" in X. This is what we need to begin topology; "nearness" lets us discuss topics like "convergence" and "continuity." For example, "f is continuous at a" means (roughly) that "if x is near a, then $f(x)$ is near $f(a)$."

The simplest way to talk about "nearness" is to equip the set X with a distance function d to tell us "how far apart" two elements of X are.

Note: As we proceed we may use some ideas taken from elementary analysis, such as the continuity of a function $f : \mathbb{R}^n \to \mathbb{R}^m$ as a source for <u>motivation</u> or <u>examples</u>, although these ideas will not be carefully defined until later in this chapter.

2. Metric and Pseudometric Spaces

Definition 2.1 Suppose $d : X \times X \to \mathbb{R}$ and that for all $x, y, z \in X$:

1) $d(x, y) \geq 0$
2) $d(x, x) = 0$
3) $d(x, y) = d(y, x)$ (<u>symmetry</u>)
4) $d(x, z) \leq d(x, y) + d(y, z)$ (the <u>triangle inequality</u>)

Such a "distance function" d is called a <u>pseudometric on X</u>. The pair (X, d) is called a <u>pseudometric space</u>. If d also satisfies

5) when $x \neq y$, then $d(x, y) > 0$

then d is called a <u>metric</u> on X and (X, d) is called a <u>metric space</u>. Of course, <u>every metric space is automatically a pseudometric space</u>.

If a pseudometric space (X, d) is not a metric space, it is because there are at least two points $x \neq y$ for which $d(x, y) = 0$. In most situations this doesn't happen; metrics come up in mathematics more often than pseudometrics. However pseudometrics <u>do</u> occasionally arise in a natural way. Moreover, many definitions and proofs actually only require using properties 1)-4). Therefore we will state our

results in terms of pseudometrics when possible. But, of course, anything we prove about pseudometric spaces is automatically true for metric spaces.

Example 2.2

1) The <u>usual metric</u> on $\mathbb{R}$ is $d(x,y) = |x-y|$. Clearly, properties 1) - 5) are true. In fact, properties 1) -5) are deliberately chosen so that a metric imitates the usual distance function.

2) The <u>usual metric</u> on $\mathbb{R}^n$ is defined as follows: if $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n)$ are in $\mathbb{R}^n$, then $d(x,y) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$. You should already know that d has properties 1) - 5). But the details to verify the triangle inequality are a little tricky, so we will go through the steps. First, we prove another useful inequality.

Suppose $a = (a_1, a_2, ..., a_n)$ and $b = (b_1, b_2, ..., b_n)$ are points in $\mathbb{R}^n$. Define

$$P(w) = \sum_{i=1}^{n}(a_i + wb_i)^2 = \sum_{i=1}^{n}a_i^2 + (2\sum_{i=1}^{n}a_ib_i)w + (\sum_{i=1}^{n}b_i^2)w^2$$

$P(w)$ is a quadratic function of w, and $P(w) \geq 0$ because $P(w)$ is a sum of squares. Therefore the equation $P(w) = 0$ has at most one real root, so it follows from the quadratic formula that

$$(2\sum_{i=1}^{n}a_ib_i)^2 - 4(\sum_{i=1}^{n}a_i^2)\,(\sum_{i=1}^{n}b_i^2) \leq 0, \qquad \text{which gives}$$

$$|\sum_{i=1}^{n}a_ib_i| \leq (\sum_{i=1}^{n}a_i^2)^{1/2}\,(\sum_{i=1}^{n}b_i^2)^{1/2}$$

This last inequality is called the <u>Cauchy-Schwarz inequality</u>. In vector notation it could be written in the form $|a \cdot b| \leq ||a|| \cdot ||b||$.

Then if $x = (x_1, x_2, ..., x_n)$, $y = (y_1, y_2, ..., y_n)$ and $z = (z_1, z_2, ..., z_n)$ are in $\mathbb{R}^n$, we can calculate

$$\begin{aligned}
d(x,z)^2 &= \sum_{i=1}^{n}(x_i - z_i)^2 = \sum_{i=1}^{n}((x_i - y_i) + (y_i - z_i))^2 \\
&= \sum_{i=1}^{n}(x_i - y_i)^2 + 2\sum_{i=1}^{n}(x_i - y_i)(y_i - z_i) + \sum_{i=1}^{n}(y_i - z_i)^2 \\
&\leq \sum_{i=1}^{n}(x_i - y_i)^2 + 2\sum_{i=1}^{n}|x_i - y_i||y_i - z_i| + \sum_{i=1}^{n}(y_i - z_i)^2 \\
&\leq \sum_{i=1}^{n}(x_i - y_i)^2 + 2(\sum_{i=1}^{n}(x_i - y_i)^2)^{1/2}(\sum_{i=1}^{n}(x_i - y_i)^2)^{1/2} + \sum_{i=1}^{n}(y_i - z_i)^2 \\
&= (d(x,y) + d(y,z))^2. \quad \text{Taking the square root of both sides gives}
\end{aligned}$$

$$d(x,z) \leq d(x,y) + d(y,z).$$

Example 2.3 We can also put other "unusual" metrics on the <u>set</u> $\mathbb{R}^n$.

1) Let d be the usual metric on $\mathbb{R}^n$ and define $d'(x,y) = 100d(x,y)$. Then d' is also a metric on $\mathbb{R}^n$. In $(\mathbb{R}^n, d')$, the "usual" distances are stretched by a factor of 100. This is just a rescaling of distances – as if we changed the units of measurement from meters to centimeters, and that change shouldn't matter in any important way. In fact, it's easy to check that if d is any metric (or pseudometric) on a set X and $\alpha > 0$, then $d' = \alpha \cdot d$ is also a metric (or pseudometric) on X.

2) If $x = (x_1, x_2, ..., x_n)$, $y = (y_1, y_2, ..., y_n)$ are points in $\mathbb{R}^n$, define

$$d_t(x,y) = \sum_{i=1}^{n} |x_i - y_i|$$

It is easy to check that d_t satisfies properties 1)-5) so $(\mathbb{R}^n, d_t)$ is a metric space. We call d_t the <u>taxicab metric</u> on $\mathbb{R}^n$. (*For $n = 2$, this means that distances are measured as if you had to move along a rectangular grid of city streets from x to y – the taxi cannot cut diagonally across a city block*).

3) If $x = (x_1, x_2, ..., x_n)$, $y = (y_1, y_2, ..., y_n)$ are points in $\mathbb{R}^n$, define

$$d^*(x,y) = \max\{|x_i - y_i| : i = 1, 2, ..., n\}$$

Then $(\mathbb{R}^n, d^*)$ is also a metric space. We will refer to d^* as the <u>max metric</u> on $\mathbb{R}^n$.

When $n = 1$, of course, d, d_t and d^* are exactly the same metric on $\mathbb{R}$.

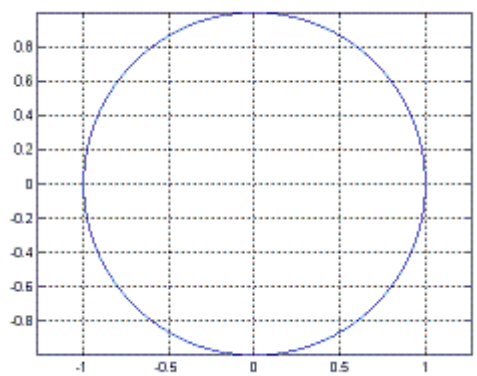
We will see later that "for topological" purposes" d', d_t, d^* are all "equivalent" metrics on $\mathbb{R}^n$. Roughly, this means that whichever of these metrics is used to measure nearness in $\mathbb{R}^n$, exactly the same functions turn out to be continuous and exactly the same sequences converge.

4) The "unit sphere" S^1 is the set of points in $\mathbb{R}^2$ that are at distance 1 from the origin. Sketch the unit sphere in $\mathbb{R}^2$ using the metrics d, d_t, d^*, and $d' = 100d$.

Since there are only two coordinates, we will write a point in $\mathbb{R}^2$ in the usual way as (x, y) rather than (x_1, x_2).

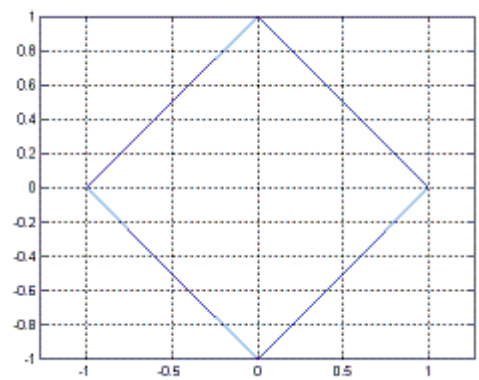
For d, we get

$$\begin{aligned} S^1 &= \{(x,y) : d((x,y),(0,0)) = 1\} \\ &= \{(x,y) : x^2 + y^2 = 1\} \end{aligned}$$



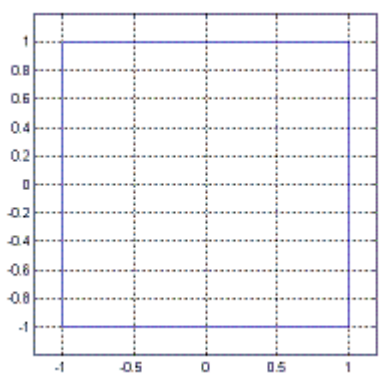
For d_t, we get

$$\begin{aligned} S^1 &= \{(x,y) : d_t((x,y),(0,0)) = 1\} \\ &= \{(x,y) : |x| + |y| = 1\} \end{aligned}$$



For d^*, we get

$$\begin{aligned} S^1 &= \{(x,y) : d^*((x,y),(0,0)) = 1\} \\ &= \{(x,y) : \max\{|x|, |y|\} = 1\} \end{aligned}$$



Of course for the metric $d' = 100d$, S^1 has the same shape as for the metric d, but the sphere is reduced in size by a scaling factor of $\frac{1}{100}$.

Switching among the metrics d, d', d_t, d^* produces unit spheres in $\mathbb{R}^n$ with different <u>sizes</u> and <u>shapes</u>. In other words, changing the metric on $\mathbb{R}^n$ may cause dramatic changes in the <u>geometry</u> of the space – for example, "areas" may change and "spheres" may no longer be "round." Changing the metric can also affect <u>smoothness</u> features of the space (spheres may turn out to have sharp corners). But it turns out, as mentioned earlier, that d, d', d_t and d^* are "equivalent" for "topological purposes." For topology, "size," "geometrical shape," and "smoothness" don't matter.

For working with $\mathbb{R}^n$, the usual metric d is the default – that is, we always assume that $\mathbb{R}^n$, or any subset of $\mathbb{R}^n$, has the usual metric d unless a different metric is explicitly stated.

Example 2.4 For each part, verify that d satisfies the properties of a pseudometric or metric.

1) For a set X, define $d(x,y) = 0$ for all $x, y \in X$. We call d the <u>trivial pseudometric</u> on X: all distances are 0. (*Under what circumstances is this d a metric?*)

2) For a set X, define $d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$. We call d the <u>discrete unit metric</u> on X.
To verify the triangle inequality: for points $x, y, z \in X$, $d(x,z) \leq d(x,y) + d(y,z)$ certainly is true if $x = z$; and if $x \neq z$, then $d(x,z) = 1$ and $d(x,y) + d(y,z) \geq 1$.

Definition 2.5 Suppose (X,d) is a pseudometric space, that $x_0 \in X$ and $\epsilon > 0$. Then $B_\epsilon(x_0) = \{x \in X : d(x,x_0) < \epsilon\}$ is called <u>the ball of radius ϵ with center at x_0</u>.
If there <u>exists</u> an $\epsilon > 0$ such that $B_\epsilon(x_0) = \{x_0\}$, then we say that x_0 is an <u>isolated point</u> in (X,d).

Example 2.6

1) In $\mathbb{R}$, $B_\epsilon(x_0)$ is the interval $(x_0 - \epsilon, x_0 + \epsilon)$. More generally, $B_\epsilon(x_0)$ in $\mathbb{R}^n$ is just the usual spherical ball with radius ϵ and center at x_0 (<u>not</u> including the boundary surface). If the metric d_t is used in $\mathbb{R}^n$, then $B_\epsilon(x_0)$ is the interior of a "diamond-shaped" region centered at x_0. (*See the earlier sketches of S^1: in $(\mathbb{R}^2, d_t)$, $B_1((0,0))$ is the region "inside" the diamond-shaped S^1.*)
In $X = [0,1]$ with the usual metric d, then $B_{\frac{1}{2}}(0) = [0, \frac{1}{2})$, $B_1(0) = [0,1)$, $B_2(0) = [0,1]$.

2) If d is the trivial pseudometric on X and $x_0 \in X$, then $B_\epsilon(x_0) = X$ for every $\epsilon > 0$.

3) If d is the discrete unit metric on X, then $B_\epsilon(x_0) = \begin{cases} \{x_0\} & \text{if } \epsilon \leq 1 \\ X & \text{if } \epsilon > 1 \end{cases}$. Therefore every point x_0 in (X,d) is isolated. If we rescale and replace d by the metric αd (where $\alpha > 0$), then it is still true that every point is isolated.

4) Let $C([0,1]) = \{f \in \mathbb{R}^{[0,1]} : f \text{ is continuous}\}$. For $f, g \in C([0,1])$, define

$$d(f,g) = \int_0^1 |f(x) - g(x)|\, dx \quad (*)$$

It is easy to check that d is a pseudometric on $C([0,1])$. In fact d is a metric: if $f \neq g$, then there must be a point $x_0 \in [0,1]$ where $|f(x_0) - g(x_0)| > 0$. By continuity, $|f(x) - g(x)| > 0$ for x's near x_0, that is, $|f(x) - g(x)| > 0$ on some interval $[a,b] \subseteq [0,1]$, where $x_0 \in [a,b]$. (*carefully explain why!*). Let $m = \min_{x \in [a,b]} |f(x) - g(x)|$ (*why does m exist?*). Then $m > 0$, so

$$d(f,g) = \int_0^1 |f(x) - g(x)|\, dx \geq \int_a^b |f(x) - g(x)|\, dx \geq \int_a^b m\, dx = m(b-a) > 0.$$

Therefore, d is a metric on $C([0,1])$.

$C([0,1])$ is a subset of the larger set $Y = \{f \in \mathbb{R}^{[0,1]} : f \text{ is integrable}\}$. We can define a distance function d on Y using the same formula (*). In this case, d is a pseudometric on Y but <u>not</u> a metric. For, example, let

$$f(x) = 0 \text{ for all } x, \text{ and } g(x) = \begin{cases} 0 & \text{if } x \neq \frac{1}{2} \\ 1 & \text{if } x = \frac{1}{2} \end{cases}$$

Then $f \neq g$ but $d(f,g) = \int_0^1 |f(x) - g(x)|\, dx = 0$

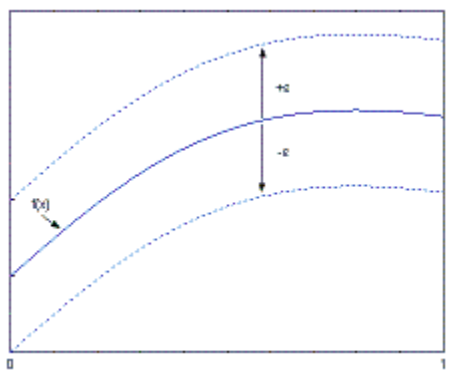
This example shows how a pseudometric that is not a metric can arise naturally in analysis.

5) On $C([0,1])$ we can also define another metric d^* by

$$\begin{aligned} d^*(f,g) &= \sup\ \{|f(x) - g(x)| : x \in [0,1]\} \\ &= \max\{|f(x) - g(x)| : x \in [0,1]\} \end{aligned}$$

(*Replacing "sup" with "max" makes sense because a theorem from analysis says that the continuous function* $|f - g|$ <u>*has*</u> *a maximum value on the closed interval* $[0,1]$.)

Then $d^*(f,g) < \epsilon$ if and only if $|f(x) - g(x)| < \epsilon$ at every point $x \in [0,1]$, so we can picture $B_\epsilon(f)$ in $(C([0,1]), d^*)$ as the set of all functions $g \in C([0,1])$ whose graph lies entirely inside a "tube of width ϵ" containing the graph of f – that is, $g \in B_\epsilon(f)$ iff g is "uniformly within ϵ of f on $[0,1]$." See the following figure.

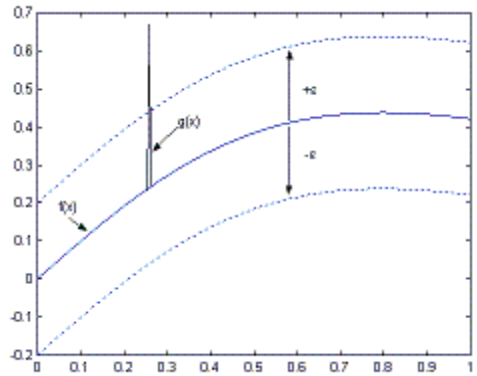


How are the metrics d and d^* from Examples 4) and 5) related? Notice that for all $f, g \in C([0,1])$:

$$d(f,g) = \int_0^1 |f(x) - g(x)|\, dx \leq \int_0^1 \max_{x\in[0,1]} |f(x) - g(x)|\, dx = \int_0^1 d^*(f,g)\, dx = d^*(f,g)\,.$$

We abbreviate this observation by writing $d \leq d^*$. It follows that $B_\epsilon^{d^*}(f) \subseteq B_\epsilon^d(f)$: so, <u>for a given</u> $\epsilon > 0$, <u>the larger metric produces the smaller ball.</u> (*Note: the superscript notation on the balls indicates which metric is being used in each case.*)

The following figure shows a function f and the graph of a function $g \in B_\epsilon^d(f) - B_\epsilon^{d^*}(f)$. The graph of g <u>coincides</u> with the graph of f, except for a tall spike: the spike takes the graph of g outside the "ϵ-tube" around the graph of f, but the spike is so thin that the $d(f,g) = \int_0^1 |f(x) - g(x)|\, dx$
$=$ "the total area between the graphs of f and g" $< \epsilon$.



6) Let $\ell_2 = \{f \in \mathbb{R}^{\mathbb{N}} : \sum_{k=1}^{\infty} f^2(k) \text{ converges}\}$. If we write $f(k) = x_k$ and use the more informal sequence notation, then $\ell_2 = \{(x_k) : x_k \in \mathbb{R} \text{ and } \sum_{k=1}^{\infty} x_k^2 \text{ converges}\}$. Thus, ℓ_2 is the set of all "square-summable" sequences of real numbers.

Suppose $x = (x_k)$ and $y = (y_k)$ are in ℓ_2 and that $a, b \in \mathbb{R}$. We claim that the sequence $ax + by = (ax_k + by_k)$ is also in ℓ_2. To see this, look at partial sums:

$$\sum_{k=1}^{n}(ax_k + by_k)^2 = a^2\sum_{k=1}^{n}x_k^2 + 2ab\sum_{k=1}^{n}x_k y_k + b^2\sum_{k=1}^{n}y_k^2 \le a^2\sum_{k=1}^{n}x_k^2 + |2a\sum_{k=1}^{n}x_k y_k| + b^2\sum_{k=1}^{n}y_k^2$$

$$\le a^2\sum_{k=1}^{n}x_k^2 + 2|a||b|\,(\sum_{k=1}^{n}x_k^2)^{1/2}\,(\sum_{k=1}^{n}y_k^2)^{1/2} + b^2\sum_{k=1}^{n}y_k^2 \quad \textit{(by the Cauchy-Schwarz inequality)}$$

$$\le a^2\sum_{k=1}^{\infty}x_k^2 + 2|a||b|(\sum_{k=1}^{\infty}x_k^2)^{1/2}(\sum_{k=1}^{\infty}y_k^2)^{1/2} + b^2\sum_{k=1}^{\infty}y_k^2 = M \in \mathbb{R} \quad \textit{(all the series converge because } x, y \in \ell_2.\textit{)}$$

Therefore the nonnegative series $\sum_{k=1}^{\infty}(ax_k + by_k)^2$ converges because it has bounded partial sums. This means that $ax + by \in \ell_2$.

In particular, if $x, y \in \ell_2$, we now know that $x - y \in \ell_2$ so $\sum_{k=1}^{\infty}(x_k - y_k)^2$ converges. Therefore it makes sense to define $d(x, y) = (\sum_{k=1}^{\infty}(x_k - y_k)^2)^{1/2}$. You should check that d is a metric on ℓ_2.

(*For the triangle inequality, notice that* $(\sum_{k=1}^{n}(x_k - z_k)^2)^{1/2} \le (\sum_{k=1}^{n}(x_k - y_k)^2)^{1/2} + (\sum_{k=1}^{n}(y_k - z_k)^2)^{1/2}$ *by the triangle inequality in* $\mathbb{R}^n$. *Letting* $n \to \infty$ *gives the triangle inequality for* ℓ_2.)

7) Suppose (X_i, d_i) are pseudometric spaces $(i = 1, ..., n)$, and that $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$ are points in the product $X = X_1 \times ... \times X_n$. Then each of the following is a pseudometric on X :

$$d(x, y) = (\sum_{i=1}^{n} d_i^2(x_i, y_i))^{1/2} \qquad\qquad d_t(x, y) = \sum_{i=1}^{n} d_i(x_i, y_i)$$

$$d^*(x, y) = \max\{d_i(x_i, y_i) : i = 1, ..., n\}$$

If each d_i is a metric, then so are d, d_t, and d^*. Notice that if each $X_i = \mathbb{R}$ and each d_i is the usual metric on $\mathbb{R}$, then d, d_t, and d^* are just the usual metric, the taxicab metric, and the max metric on $\mathbb{R}^n$. As we shall see, it turns out that these metrics on X are all equivalent for "topological purposes."

Definition 2.7 Suppose (X, d) is a pseudometric space and $O \subseteq X$. We say that <u>O is open in (X, d)</u> if for each $x \in O$ there is an $\epsilon > 0$ such that $B_\epsilon(x) \subseteq O$. (*Of course, ϵ may depend on x.*)

For example,

i) The sets $\emptyset$ and X are open in any space (X, d).

ii) The intervals (a, b), $(-\infty, a)$, (b, ∞), and $(-\infty, \infty) = \mathbb{R}$ are open in $\mathbb{R}$. (*Fortunately this terminology is consistent with the fact that these intervals are called "open intervals" in calculus books.*)

But notice that the interval (a, b), when viewed as a subset of the x-axis in $\mathbb{R}^2$, is <u>not</u> open in $\mathbb{R}^2$. Similarly, $\mathbb{R}$ is an open set in $\mathbb{R}$, but $\mathbb{R}$ (viewed as the x-axis) is <u>not</u> open in $\mathbb{R}^2$.

iii) The intervals $[a, b]$, $[a, b)$ and $(a, b]$ are not open in $\mathbb{R}$. But the sets $[a, b)$ and $[a, b]$ are open in the metric space $([a, b], d)$.

Examples ii) and iii) illustrate that "open" is not a property that depends just on the set A: whether or not a set A is open depends on the larger space in which it "lives" – that is, "open" is a relative term.

The next theorem tells us that the <u>balls in (X, d) are "building blocks" from which all open sets can be constructed</u>.

Theorem 2.8 A set $O \subseteq X$ is open in (X, d) if and only if O is a union of a collection of balls.

Proof If O is open, then for each $x \in O$ there is an $\epsilon_x > 0$ such that $B_{\epsilon_x}(x) \subseteq O$ and therefore we can write $O = \bigcup_{x \in O} B_{\epsilon_x}(x)$.

Conversely, suppose $O = \bigcup_{x \in C} B_{\epsilon_x}(x)$ for some indexing set $C \subseteq O$. We must show that if $y \in O$, then $B_\epsilon(y) \subseteq O$ for some $\epsilon > 0$. Since $y \in O$, we know that $y \in B_{\epsilon_{x_0}}(x_0)$ for some $x_0 \in C$. Then $d(x_0, y) = \delta < \epsilon_{x_0}$. Let $\epsilon = \frac{1}{2}(\epsilon_{x_0} - \delta) > 0$ and consider $B_\epsilon(y)$. If $z \in B_\epsilon(y)$, then $d(z, x_0) \leq d(z, y) + d(y, x_0) < \epsilon + \delta = \frac{1}{2}(\epsilon_{x_0} - \delta) + \delta = \frac{1}{2}\epsilon_{x_0} + \frac{1}{2}\delta < \frac{1}{2}\epsilon_{x_0} + \frac{1}{2}\epsilon_{x_0} = \epsilon_{x_0}$, so $z \in B_{\epsilon_{x_0}}(x_0)$. Therefore $B_\epsilon(y) \subseteq B_{\epsilon_{x_0}}(x_0) \subseteq O$. •

Corollary 2.9 a) Each ball $B_\epsilon(x)$ is open in (X, d).

b) A point x_0 in a pseudometric space (X, d) is isolated iff $\{x_0\}$ is an open set.

Definition 2.10 Suppose (X, d) is a pseudometric space. The <u>topology</u> $\mathcal{T}_d$ <u>generated</u> <u>by d</u> is the collection of all open sets in (X, d). In other words, $\mathcal{T}_d = \{O : O \text{ is open in } (X, d)\} = \{O : O \text{ is a union of balls}\}$.

Theorem 2.11 Let $\mathcal{T}_d$ be the topology in (X, d). Then

i) $\emptyset, X \in \mathcal{T}_d$
ii) if $O_\alpha \in \mathcal{T}_d$ for each $\alpha \in A$, then $\bigcup_{a \in A} O_a \in \mathcal{T}_d$
iii) if $O_1, ..., O_n \in \mathcal{T}_d$, then $O_1 \cap ... \cap O_n \in \mathcal{T}_d$.

(Conditions ii) and iii) say that the collection $\mathcal{T}_d$ is "closed under unions" and "closed under finite intersections.")

Proof $\emptyset$ is the union of the empty collection of open balls, and $X = \bigcup_{x \in X} B_1(x)$, so $\emptyset, X \in \mathcal{T}_d$.

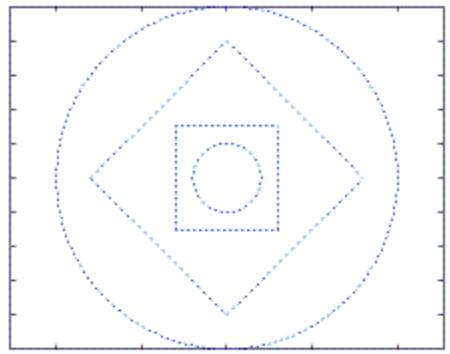
Suppose $x \in O = \bigcup_{\alpha \in A} O_\alpha$ where each $O_\alpha \in \mathcal{T}_d$. Then x is in one of these open sets, say O_{α_0}. So for some $\epsilon > 0$, $x \in B_\epsilon(x) \subseteq O_{\alpha_0} \subseteq O$. Therefore O is open, that is, $O \in \mathcal{T}_d$.

To verify iii), suppose $O_1, O_2, \dots, O_n \in \mathcal{T}_d$ and that $x \in O_1 \cap O_2 \cap ... \cap O_n$. For each $i = 1, ..., n$, there is an $\epsilon_i > 0$ such that $x \in B_{\epsilon_i}(x) \subseteq O_i$. Let $\epsilon = \min\{\epsilon_1, \epsilon_2, ..., \epsilon_n\} > 0$. Then $B_\epsilon(x) \subseteq O_1 \cap O_2 \cap ... \cap O_n$. Therefore $O_1 \cap O_2 \cap ... \cap O_n \in \mathcal{T}_d$. •

Example 2.12 The set $O_n = (-\frac{1}{n}, \frac{1}{n})$ is open in $\mathbb{R}$ for every $n \in \mathbb{N}$. However, $\bigcap_{n=1}^{\infty} O_n = \{0\}$ is <u>not</u> open in $\mathbb{R}$: so an intersection of <u>infinitely many</u> open sets might not be open. (*Where does the proof for part iii) in Theorem 2.11 break down if we intersect infinitely many open sets?*)

Notice that <u>different pseudometrics can produce the same topology</u> on a set X. For example, if d is a metric on X and we set $d' = 2d$, then d and d' produce the same collection of balls (with radii measured differently): for each $\epsilon > 0$, the ball $B_\epsilon^d(x)$ is the same set as the ball $B_{2\epsilon}^{d'}(x)$. If we get the same balls from each metric, then we must also get the same open sets: $\mathcal{T}_d = \mathcal{T}_{d'}$ (*see Theorem 2.8*).

We can see a less trivial example in $\mathbb{R}^2$. Let d, d_t, and d^* be the usual metric, the taxicab metric, and the max metric on $\mathbb{R}^2$. Clearly any set which is a union of d-balls (or d^* balls) can also be written as a union of d_t-balls, and vice-versa. (*Explain why! See the following picture for* $\mathbb{R}^2$.)



Therefore all three metrics produce the same topology: $\mathcal{T}_d = \mathcal{T}_{d_t} = \mathcal{T}_{d^*}$ even though the balls are different for each metric. It turns out that <u>the open sets in (X, d) are the most important objects from a topological point of view</u>, so in that sense these metrics are all equivalent. (As mentioned earlier, these metrics do change the "shape" and "smoothness" of the balls and therefore these metrics are not equivalent geometrically.)

Definition 2.13 Suppose d and d' are two pseudometrics (or metrics) on a set X. We say that d and d' are <u>equivalent</u> (written $d \sim d'$) if $\mathcal{T}_d = \mathcal{T}_{d'}$, that is, if d and d' generate the same collection of open sets.

Example 2.14

1) If d is the discrete unit metric on X, then each singleton set $\{x\} = B_1(x)$ is a ball, so each $\{x\}$ is open – equivalently, every point x is isolated in (X, d). If $A \subseteq X$, then $A = \bigcup_{a \in A} \{a\}$ is open because A is a union of balls. Therefore $\mathcal{T}_d = \mathcal{P}(X)$, called the <u>discrete topology on X</u>. This is the <u>largest</u> possible topology on X.

$$\text{If } d'(x, y) = \begin{cases} 17 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases} \text{ on a set } X, \text{ then } d' \sim d, \text{ where } d \text{ is the discrete unit metric.}$$

More generally, $\alpha d \sim d$ for any $\alpha > 0.$, and all of these metrics generate the discrete topology.

2) Let $X = \{ \frac{1}{n} : n \in \mathbb{N} \}$. Let d be the usual metric on X and let d' be the discrete unit metric on X. For each n, $B_\epsilon^d(\frac{1}{n}) = \{\frac{1}{n}\}$ if we choose a sufficiently small ϵ. Therefore, just as in part 1), every subset of X is open in (X, d). But every subset in (X, d') also is open, so $d \sim d'$ (even though d and d' are not constant multiples of each other).

3) Let d be the trivial pseudometric on a set X. Are there any other pseudometrics d' on X for which $d' \sim d$?

3. The topology of $\mathbb{R}$

What do the open sets in $\mathbb{R}$ look like? Since ϵ-balls in $\mathbb{R}$ are intervals of the form $(a - \epsilon, a + \epsilon)$, the open sets are precisely the sets which are unions of open intervals. But we can actually say more to make the situation even clearer. We begin by making a precise definition for the term "interval."

Definition 3.1 A subset I of $\mathbb{R}$ is <u>convex</u> if whenever $x \le y \le z$ and $x, z \in I$, then $y \in I$. A convex subset of $\mathbb{R}$ is called an <u>interval</u>.

It is easy to give examples of intervals in $\mathbb{R}$. The following theorem states that the obvious examples are the only examples.

Theorem 3.2 $I \subseteq \mathbb{R}$ is an interval iff I has one of the following forms (where $a < b$):

$$(-\infty, \infty),\ (-\infty, a),\ (-\infty, a],\ [a, \infty),\ (a, \infty),\ (a, b),\ [a, b),\ (a, b],\ [a, b],\ \{a\},\ \emptyset \qquad (*)$$

Proof It is clear that each of the sets in the list is convex and therefore each is an interval.

Conversely, we need to show that every interval I has one of these forms. Clearly, if $|I| \le 1$, then $I = \emptyset$ or $I = \{a\}$. If $|I| \ge 2$ then the definition of interval implies that I must be infinite. The remainder of the proof uses the completeness property (= "least upper bound property") of $\mathbb{R}$, and the argument falls into several cases:

Case I: I is bounded both above and below. Then I has a least upper bound and a greatest lower bound: let $a = \inf I$ and $b = \sup I$. Of course a and b might or might not be <u>in</u> I.

a) if $a, b \in I$,	we claim $I = [a, b]$
b) if $a \in I$ but $b \notin I$,	we claim $I = [a, b)$
c) if $a \notin I$ but $b \in I$,	we claim $I = (a, b]$
d) if $a, b \notin I$,	we claim $I = (a, b)$

Case II: I is bounded below but not above. In this case, let $a = \inf I$.

a) if $a \in I$,	we claim $I = [a, \infty)$
b) if $a \notin I$,	we claim $I = (a, \infty)$

Case III: I is bounded above but not below. In this case, let $b = \sup I$.

a) if $b \in I$,	we claim $I = (-\infty, b]$
b) if $b \notin I$,	we claim $I = (-\infty, b)$

Case IV: I is not bounded above or below. In this case, we claim $I = (-\infty, \infty)$.

In all cases, the proofs are similar and use properties of sups and infs. To illustrate, we prove case Ic):

> If $x \in I$, then $x \le \sup I = b$. Also, $x \ge \inf I = a$, and because $a \notin I$, we get $x > a$. So $I \subseteq (a, b]$.
>
> We still need to show that $(a, b] \subseteq I$, so suppose $x \in (a, b]$. Then $x > a = \inf I$, so x is not a lower bound for I. This means that there is a point $z \in I$ such that $z < x$. Then $z < x \le b$ where $z, b \in I$, and I is convex, so $x \in I$. Therefore $(a, b] \subseteq I$, so $I = (a, b]$. •

Note: We used the completeness property to prove Theorem 3.2. In fact, Theorem 3.2 is <u>*equivalent*</u> *to the completeness property. To see this* :

> *Assume Theorem 3.2 is true and that A is a nonempty subset of $\mathbb{R}$ that has an upper bound. Let $I = \{x \in \mathbb{R} : x \leq c$ for some $c \in A\}$. Then I is an interval (suppose $x \leq y \leq z$ where $x, z \in I$. Then $z \leq c$ for some $c \in A$; therefore $y \leq c$ so, by definition of I, $y \in I$.).*
>
> *Since $A \neq \emptyset$, I must be infinite. An upper bound for A must also be an upper bound for I. Since I is an interval with an upper bound, I must have one of the forms $(-\infty, b]$, $(-\infty, b)$, (a,b), $[a, b)$, $(a, b]$, $[a, b]$. Then it's not hard to check that A has a least upper bound, namely sup $A = b$.*

This is an observation I owe to Professor Robert McDowell.

It is clear that an intersection of intervals in $\mathbb{R}$ is an interval (*why*?). But a union of intervals might not be an interval: for example, $[0, 1] \cup [2, 3]$. However, if every pair of intervals in a collection "overlap," then the union <u>is</u> an interval. The following theorem makes this precise.

Theorem 3.3 Suppose $\mathcal{I}$ is a collection of intervals in $\mathbb{R}$ and that for every pair $I, J \in \mathcal{I}$, we have $I \cap J \neq \emptyset$. Then $\bigcup \mathcal{I}$ is an interval. In particular, if $\bigcap \mathcal{I} \neq \emptyset$, then $\bigcup \mathcal{I}$ is an interval.

Proof Let $a, b \in \bigcup \mathcal{I}$ and suppose that $a \leq x \leq b$. We need to show that $x \in \bigcup \mathcal{I}$.

Pick intervals I, J in $\mathcal{I}$ with $a \in I$ and $b \in J$, and choose a point $z \in I \cap J$. If $x = z$, then $x \in \bigcup \mathcal{I}$ and we are done. Otherwise, either $z < x \leq b$ or $a \leq x < z$. Therefore <u>either</u> x is either between two points of J and so $x \in J$; <u>or</u> x is between two points of I, so $x \in I$. Either way, we conclude that $x \in \bigcup \mathcal{I}$. •

We can now give a more careful description of the open sets in $\mathbb{R}$.

Theorem 3.4 Suppose $O \subseteq \mathbb{R}$. O is open in $\mathbb{R}$ if and only if O is the union of a <u>countable</u> collection of <u>pairwise disjoint</u> open intervals.

Proof ($\Leftarrow$) Open intervals in $\mathbb{R}$ are open sets, and a union of any collection of open sets is open.

($\Rightarrow$) Suppose O is open in $\mathbb{R}$. For each $x \in O$, there is an open interval (ball) I such that $x \in I \subseteq O$. Let $G_x = \bigcup\{I : I$ is an open interval and $x \in I \subseteq O\}$. Then $x \in G_x \subseteq O$.
By Theorem 3.3, G_x is also an open interval (*in fact, G_x is the* <u>*largest*</u> *open interval containing x and inside O; why?*). It is easy to see that there can be distinct points $x, y \in O$ for which $G_x = G_y$. In fact, we claim that if $x, y \in O$, then either $G_x = G_y$ or $G_x \cap G_y = \emptyset$.

> If $G_x \cap G_y \neq \emptyset$, then there is a point $z \in G_x \cap G_y$. By Theorem 3.3, $G_x \cup G_y$ is an open interval, a subset of O, and containing both x and y. Therefore $G_x \cup G_y$ is a set in the collection whose union is G_x. Therefore $G_x \cup G_y \subseteq G_x$. Similarly, $G_x \cup G_y \subseteq G_y$, so $G_x = G_y$.

Removing any repetitions, we let $\mathcal{D}$ be the collection of the <u>distinct</u> intervals G_x that arise in this way. Clearly, $O = \cup \mathcal{D}$ and $\mathcal{D}$ is countable because the members of $\mathcal{D}$ are pairwise disjoint:

> for each $I \in \mathcal{D}$, we can pick a rational number $q_I \in I$, and these q_I's are distinct. So there can be no more $\mathcal{I}$'s in $\mathcal{D}$ than there are rational numbers. (*More formally, the function* $f : \mathcal{D} \to \mathbb{Q}$ *given by* $f(I) = q_I$ *is one-to-one.*) •

We are now able to count the open sets in $\mathbb{R}$.

Corollary 3.5 There are exactly c open sets in $\mathbb{R}$.

Proof Let $\mathcal{T}_d$ be the usual topology on $\mathbb{R}$. We want to prove that $|\mathcal{T}_d| = c$.

For each $r \in \mathbb{R}$, the interval $(-\infty, r) \in \mathcal{T}_d$, so $|\mathcal{T}_d| \geq c$.

Let $\mathcal{I}$ be the set of <u>all</u> open intervals in $\mathbb{R}$. Then $|\mathcal{I}| = c$ (*why?*). For each $O \in \mathcal{T}_d$, <u>pick</u> a sequence $I_1, I_2, ...I_n, ... \in \mathcal{I}$ for which $O = \bigcup_{n=1}^{\infty} I_n$. (*We could also choose the* I_n*'s to be pairwise disjoint, but that is unnecessary in this argument – the important thing here is that there are only countably many* I_n*'s.*) Then we have a function $f : \mathcal{T}_d \to \mathcal{I}^{\mathbb{N}}$ given by $f(O) = (I_1, I_2, ..., I_n, ...)$. The function f is clearly one-to-one, so $|\mathcal{T}| \leq |\mathcal{I}^{\mathbb{N}}| = c^{\aleph_0} = c$. •

Exercises

E1. The following two statements refer to a metric space (X, d). Either prove or give a counterexample for each statement. (*These statements illustrate the danger of assuming that familiar features of $\mathbb{R}^n$ necessarily carry over to arbitrary pseudometric spaces.*)

a) $B_\epsilon(x) = B_\epsilon(y)$ implies $x = y$ (i.e., "a ball can't have two centers")

b) The diameter of $B_\epsilon(x)$ must be bigger than ϵ. (*The diameter of a set A in a metric space is defined to be sup* $\{d(x, y) : x, y \in A\} \leq \infty$.)

E2. The "taxicab" metric on $\mathbb{R}^2$ is defined by $d_t((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|$. Draw the set of points in $(\mathbb{R}^2, d_t)$ that are equidistant from $(0, 0)$ and $(3, 4)$.

E3. Suppose (X, d) is a metric space.

$a)$ Define $d^*(x, y) = \min\{1, d(x, y)\}$. Prove that d^* is also a metric on X, and that $\mathcal{T}_d = \mathcal{T}_{d^*}$.

b) Define $d^{**}(x, y) = \frac{d(x,y)}{1+d(x,y)}$. Prove that d^{**} is also a metric on X and that $\mathcal{T}_d = \mathcal{T}_{d^{**}}$

Hint: Let d *be a metric on X and suppose f is a function from the nonnegative real numbers to the nonnegative real numbers for which:* $f(0) = 0$, $x \leq y \Rightarrow f(x) \leq f(y)$, *and* $f(x + y) \leq f(x) + f(y)$ *for all nonnegative* x, y. *Prove that* $d'(x, y) = f(d(x, y))$ *is also a metric on* X. *Then consider the particular function* $f(x) = \frac{x}{1+x}$.

<u>Note</u>: For all x, y in X, $d^*(x, y) \leq 1$ and $d^{**}(x, y) \leq 1$ – that is, d^* and d^{**} are <u>bounded</u> metrics on X. Thus any metric d on X can be replaced by an equivalent bounded metric – that is, a bounded metric that generates the same topology. "Boundedness" is a <u>property determined by the particular metric, not by the topology</u>.

E4. Suppose $f : \mathbb{R} \to \mathbb{R}$. Let d be the usual metric on $\mathbb{R}$, and d' the usual metric on $\mathbb{R}^2$. Define a new distance function d'' on $\mathbb{R}$ by $d''(x, y) = d'((x, f(x)), (y, f(y)))$. Prove that d'' is a metric on $\mathbb{R}$.

Must d'' be equivalent to d ? If not, can you (*precisely, or informally*) describe conditions which will guarantee that $d'' \sim d$?

E5. Suppose a function $d : X \times X \to \mathbb{R}$ satisfies conditions 1), 2), 3), and 5) in the definition of a metric, but that the triangle inequality is replaced by: $x, y, z \in X$

$$d(x, z) \geq d(x, y) + d(y, z).$$

Prove that $|X| \leq 1$.

E6. Suppose A is a finite open set in a metric space (X, d). Prove that every point of A is isolated in (X, d).

E7. Suppose (X, d) is a metric space and $x \in X$. Prove that the following two statements are <u>equivalent</u>:

i) x is not an isolated point of X

ii) every open set that contains x is infinite.

E8. The definition of an open set in (X, d) reads: O is open if for all $x \in O$, there is an $\epsilon > 0$ such that $B_\epsilon(x) \subseteq O$. In this definition, ϵ may depend on x.

Suppose we define O to be <u>uniformly open</u> if there is an $\epsilon > 0$ such that for all $x \in O$, $B_\epsilon(x) \subseteq O$ – that is, the same ϵ works for every $x \in O$. (*"Uniformly open" is <u>not</u> a standard term.*)

a) What are the uniformly open subsets of $\mathbb{R}^n$?
b) What are the uniformly open sets in (X, d) if d is the trivial pseudometric?
c) What are the uniformly open sets in (X, d) if d is the discrete unit metric?

E9. Let p be a fixed prime number. We define the <u>p-adic absolute value</u> $|\ |_p$ (sometimes called the p-adic norm) on the set of rational numbers $\mathbb{Q}$ as follows:

If $0 \neq x \in \mathbb{Q}$, write $x = \frac{p^k m}{n}$ for integers *k,m,n*, where p does not divide m or n, and define $|x|_p = p^{-k} = \frac{1}{p^k}$. (*Of course, k may be negative.*). Also, define $|0|_p = 0$.

Prove that $|\ |_p$ "behaves the way an absolute value (norm) should" – that is, for all $x, y \in \mathbb{Q}$

a) $|x|_p \geq 0$ and $|x|_p = 0$ iff $x = 0$
b) $|xy|_p = |x|_p \cdot |y|_p$
c) $|x + y|_p \leq |x|_p + |y|_p$

$|\ |_p$ actually satisfies an inequality <u>stronger</u> than the one in part c). Prove that

d) $|x + y|_p \leq \max\{|x|_p, |y|_p\} \leq |x|_p + |y|_p$

Whenever we have an absolute value (norm), we can use it to define a distance function:

for $x, y \in \mathbb{Q}$, let $d_p(x, y) = |x - y|_p$

e) Prove that d_p is a metric on $\mathbb{Q}$, and show that d_p actually satisfies an inequality <u>stronger</u> than the usual triangle inequality, namely:

for all $x, y, z \in \mathbb{Q}$, $d_p(x, z) \leq \max\{d_p(x, y), d_p(y, z)\}$

f) Give a specific example for x, y, z, p for which

$d_p(x, z) < \max\{d_p(x, y), d_p(y, z\})$

(*Hint: It might be convenient to be able to refer to the exponent "k" associated with a particular x. If $x = \frac{p^k m}{n}$, then k roughly refers to the "number of p's that can be factored out of x" – so we can call $k = \nu(x)$. Prove that $\nu(a - b) \geq \min\{\nu(a), \nu(b)\}$ whenever $a,b \in \mathbb{Q}$ with $a,b \neq 0$ and $a \neq b$. Note that strict inequality can occur here: for example, when $p = 3$, $\nu(8) = \nu(2) = 0$, but $\nu(8 - 2) = \nu(6) = 1$.*)

g) Suppose $p = 2$. Calculate $d_2(2^n, 0)$. What are $\lim_{n\to\infty} d_2(2^n, 0)$ and $\lim_{n\to\infty} d_2(4^n, 0)$?

4. Closed Sets and Operators on Sets

Definition 4.1 Suppose (X, d) is a pseudometric space and that $F \subseteq X$. We say that F is <u>closed in</u> (X, d) if $X - F$ is open in (X, d).

From the definitions:

F is closed in (X, d)
iff $X - F$ is open in (X, d)
iff for all $x \in X - F$, there is an $\epsilon > 0$ for which $B_\epsilon(x) \subseteq X - F$
iff for all $x \in X - F$, there is an $\epsilon > 0$ for which $B_\epsilon(x) \cap F = \emptyset$.

The open sets in X completely determine the closed sets, and vice-versa. This means that, in some sense, the topology $\mathcal{T}_d$ (the collection of open sets) and the collection of closed sets contain exactly the same "information" about a space (X, d).

The close connection between the closed sets and the open sets is reflected in the following theorem.

Theorem 4.2 In any pseudometric space (X, d),

i) $\emptyset$ and X are closed
ii) if F_α is closed for each $\alpha \in A$, then $\bigcap_{\alpha \in A} F_\alpha$ is closed
("an intersection of closed sets is closed")
iii) if $F_1, ..., F_n$ are closed, then $\bigcup_{i=i}^{n} F_i$ is closed
("a finite union of closed sets is closed")

Proof These statements follow from the corresponding properties of open sets just by taking complements. Since $\emptyset$ and X are open, their complements $X - \emptyset = X$ and $X - X = \emptyset$ are closed.

Suppose F_α is closed for each $\alpha \in A$. Then each set $X - F_\alpha$ is open, so $\bigcup_{\alpha \in A}(X - F_\alpha)$ is open, and therefore its complement $X - (\bigcup_{\alpha \in A}(X - F_\alpha)) = \bigcap_{\alpha \in A} X - (X - F_\alpha) = \bigcap_{\alpha \in A} F_\alpha$ is closed.

The proof of part iii) is similar to that for part ii) and uses the fact that a finite <u>intersection</u> of open sets is open. •

Exercise: Give an example to show that an <u>infinite</u> union of closed sets might not be closed.

Example 4.3

1) The interval $[0, 1]$ is closed in $\mathbb{R}$ because its complement $\mathbb{R} - [0, 1] = (-\infty, 0) \cup (1, \infty)$ is open. Equivalently, we could say that $[0, 1]$ is closed because for each $x \notin [0, 1]$, there is an $\epsilon > 0$ for which $B_\epsilon(x) \cap [0, 1] = (x - \epsilon, x + \epsilon) \cap [0, 1] = \emptyset$.

2) A set might be <u>neither</u> open <u>nor</u> closed: as examples, consider the following subsets of $\mathbb{R}$:

$[0, 1)$ $\qquad$ $\mathbb{Q}$ $\qquad$ $\mathbb{P}$

3) A set can be <u>both</u> open and closed – that is, these terms are <u>not mutually exclusive</u>. Such sets in (X, d) are called <u>clopen</u> sets. For example, $\emptyset$ and X are clopen in every pseudometric space (X, d). Sometimes X contains other clopen sets and sometimes not. For example:

a) in $\mathbb{R}$, for example, $\emptyset$ and $\mathbb{R}$ are the <u>only</u> clopen subsets. (*This fact is not too hard to prove but it is also not obvious – the proof depends on the completeness property* (= *"least upper bound property"*) *in* $\mathbb{R}$. *We will prove this fact later when we need it in Chapter V.*)

b) in the space $X = [0, 1] \cup [3, 4]$ with the usual metric d, the set $[0, 1]$ is clopen.

4) In any pseudometric space (X, d), the set $F = \{x \in X : d(a, x) \leq \epsilon\}$ is a closed set. To see this, suppose $y \notin F$, Then $d(a, y) = \delta > \epsilon$. Let $\epsilon_1 = \delta - \epsilon > 0$. Then $B_{\epsilon_1}(y) \cap F = \emptyset$.
(If $z \in B_{\epsilon_1}(y) \cap F$, then we would have $d(a, y) \leq d(a, z) + d(z, y) < \epsilon + \epsilon_1 = \epsilon + (\delta - \epsilon) = \delta$, which is false.)

$\{x \in X : d(a, x) \leq \epsilon\}$ is called the <u>closed ball centered at x with radius ϵ.</u>

For example, $[0, 1] = \{x \in \mathbb{R} : d(\frac{1}{2}, x) = |x - \frac{1}{2}| \leq \frac{1}{2}\}$ is the "closed ball" centered at $\frac{1}{2}$.

5) Let $X = [0, 1] \cup [2, 5)$ with the usual metric d.

Both $[0, 1]$ and $[2, 5)$ are clopen sets in X
but $[3, 4)$ is neither open nor closed in X

Notice again that "open" and "closed" are not absolute terms: whether a set A is open (or closed) is relative to the larger space X that contains A.

6) Let d be the discrete unit metric. Then $\mathcal{T}_d$ is the <u>discrete topology</u> and every subset of X is open. Then every subset of X is clopen.

7) Let d be the trivial pseudometric on X. For every $\epsilon > 0$ and every $x \in X$, the ball $B_\epsilon(x) = X$. In this space, a union of balls must be either X or $\emptyset$ (for the union of an empty collection of balls). Therefore $\mathcal{T}_d = \{\emptyset, X\}$. This is called the <u>trivial topology</u> on X.

Since $\emptyset$ and X must be open in any space (X, d), the trivial topology is the <u>smallest</u> possible topology on X. In (X, d), the only closed sets are $\emptyset$ and X.

Using open and closed sets in (X, d), we can define some useful "operators" on subsets of X. An "operator" creates a new subset of X from an old one.

Definition 4.4 Suppose (X, d) is a pseudometric space and $A \subseteq X$.

the <u>interior of A in X</u>	$= \text{int}_X A$	$= \bigcup\{O : O \text{ is open and } O \subseteq A\}$
the <u>closure of A in X</u>	$= \text{cl}_X A$	$= \bigcap\{F : F \text{ is closed and } F \supseteq A\}$
the <u>frontier (or boundary) of A in X</u>	$= \text{Fr}_X A$	$= \text{cl}_X A \cap \text{cl}_X(X - A)$

We will omit the subscript "X" when the context makes clear the space X in which the operations are being performed. Sometimes int A and cl A are denoted A° and $\bar{A}$ respectively. Some books use the notation ∂A for Fr A, but the symbol ∂A has a different meaning in algebraic topology so we will avoid using it here.

Theorem 4.5 Suppose (X, d) is a pseudometric space and that $A \subseteq X$. Then

1) a) int A is the largest open subset of A (that is, if O is open and $O \subseteq A$, then $O \subseteq$ int A).
   b) A is open iff $A =$ int A (since int $A \subseteq A$, the equality is equivalent to $A \subseteq$ int A).
   c) $x \in$ int A iff there is an open set O such that $x \in O \subseteq A$
   iff $\exists\, \epsilon > 0$ such that $B_\epsilon(x) \subseteq A$.

   Informally, we can think of 1c) as saying that the interior of A consist of those points "comfortably inside" A ("surrounded by a small cushion"). These are the points not "on the boundary of A."

2) a) cl A is the smallest closed set that contains A (that is, if F is closed and $F \supseteq A$, then $F \supseteq$ cl A)
   b) A is closed iff $A =$ cl A (since $A \subseteq$ cl A, the equality is equivalent to cl $A \subseteq A$).
   c) $x \in$ cl A iff for every open set O containing x, $O \cap A \neq \emptyset$
   iff for every $\epsilon > 0$, $B_\epsilon(x) \cap A \neq \emptyset$.

   Informally, 2c) states that cl$A$ consists of the points in X that can be approximated arbitrarily closely by points from within the set A.

3) a) Fr A is closed and Fr $A =$ Fr $(X - A)$.
   b) A is clopen iff Fr $A = \emptyset$.
   c) $x \in$ Fr A iff for every open set O containing x, $O \cap A \neq \emptyset$ and $O \cap (X - A$

   Informally, 3c) states that Fr A consists of those points in X that can be approximated arbitrarily closely both by points from within A and by points from outside A.

Proof 1) a) int A is a union of open sets, so it is open and, by definition int $A \subseteq A$.
If O is open and $O \subseteq A$, then O is one of the sets whose union gives int A, so $O \subseteq$ int A.

b) If $A =$ int A, then A is open.
Conversely, if A is open, then A is clearly the largest open subset of A, so $A =$ int A.

c) Since int A is a union of open sets, it is clear that $x \in$ int A iff $x \in O \subseteq A$ for some open set O. Since a open set is open iff it is a union of ϵ-balls, the remainder of the assertion is obviously true.

2) Exercise

3) a) Fr A is closed because it is an intersection of two closed sets, and

$$\text{Fr}\,(X-A) = \text{cl}(X-A) \cap \text{cl}(X-(X-A)) = \text{cl}(X-A) \cap \text{cl}(A) = \text{Fr}\,A.$$

b) If A is clopen, then so is $X-A$. Therefore $\text{Fr}(A) = \text{cl}\,(A) \cap \text{cl}(X-A)$ $= A \cap (X-A) = \emptyset$.

Conversely, if $\text{Fr}\,A = \text{cl}\,(A) \cap \text{cl}(X-A) = \emptyset$, then $\text{cl}\,A \subseteq X - \text{cl}(X-A)$ $\subseteq X-(X-A) = A$, so A is closed.

Similarly we show that $X-A$ is closed, so A is clopen.

c) $x \in \text{Fr}\,A$ iff x in both $\text{cl}\,A$ and $\text{cl}\,(X-A)$. By 2c), this is true iff each open set containing x intersects both A and $X-A$.

Since an open set is a union of ϵ-balls, the remainder of the assertion is clearly true. •

Notice that in each part of Theorem 4.5, item c) gives you a criterion <u>that uses only the open sets</u> (not ϵ-balls!) to decide whether or not $x \in \text{int}\,A$, $x \in \text{cl}\,A$, or $x \in \text{Fr}\,A$. *This is important!* It means that if we change the d to an <u>equivalent</u> pseudometric d', then $\text{int}\,A$, $\text{cl}\,A$, and $\text{Fr}\,A$ do <u>not</u> change, since d and d' produce the same open sets. In other words, we can say that int, cl, and Fr are <u>topological</u> operators: they depend on only the topology, and not on the particular metric that produces the topology. For example if $A \subseteq \mathbb{R}^n$, then A will have the same interior, same closure, and same frontier whether we measure distances using the usual metric d, the taxicab metric d_t, or the max-metric d^*.

Example 4.6 (*Be sure you understand each statement!*)

1) In $\mathbb{R}$:

$$\text{int}\,\mathbb{R} = \mathbb{R} \qquad \text{cl}\,\mathbb{R} = \mathbb{R} \qquad \text{Fr}\,\mathbb{R} = \emptyset$$
$$\text{int}\,\mathbb{Q} = \emptyset \qquad \text{cl}\,\mathbb{Q} = \mathbb{R} \qquad \text{Fr}\,\mathbb{Q} = \mathbb{R}$$

$$\text{int}\,[0,1) = (0,1) \qquad \text{cl}\,[0,1) = [0,1]$$
$$\text{Fr}\,[0,1) = \{0,1\} \qquad \text{Fr}\,(\text{Fr}\,\mathbb{Q}) = \text{Fr}\,(\mathbb{R}) = \emptyset.$$

In any space (X,d), it is obviously true that $\text{int}(\text{int}\,A) = \text{int}\,A$ and $\text{cl}(\text{cl}\,A) = \text{cl}\,A$. But this need not be true for Fr, as the last example shows. (*It <u>is</u> true that* $Fr(Fr(Fr\,A)) = Fr(Fr\,A)$ *in any space* X. *However, this is not a useful fact, and it is also not very interesting to prove.*)

2) $X = [0,2)$ (with the usual metric)

$$\text{cl}_X\,[0,1) = [0,1] \qquad \text{int}_X[0,1) = [0,1)$$
$$\text{cl}_X\,[1,2) = [1,2) \qquad \text{int}_X[1,2) = (1,2)$$

$\text{Fr}_X\,[0,1) = [0,1] \cap [1,2) = \{1\}$. *Note that* $0 \notin Fr_X[0,1)$ *because* 0 *cannot be "approximated arbitrarily closely" by points from* $X - [0,1)$.

3) Suppose d is the discrete unit metric on X. If $A \subseteq X$, then A is clopen so we get $\text{cl}\,A = A$, $\text{int}\,A = A$, and $\text{Fr}\,A = \emptyset$.

On the other hand, d is the trivial pseudometric on X. If A is any nonempty, proper subset of X, then $\text{cl}\,A = X$, $\text{int}\,A = \emptyset$, and $\text{Fr}\,A = X$.

4) In (ℓ_2, d), let A be the set of sequences with all terms rational:

$$A = \{x = (x_i) \in \ell_2 : \forall i,\ x_i \in \mathbb{Q}\} = \mathbb{Q}^{\mathbb{N}} \cap \ell_2.$$

<u>We claim</u> that cl $A = \ell_2$ – in other words, that any point $y = (y_i) \in \ell_2$ can be approximated arbitrarily closely by a point from A. So let $\epsilon > 0$. We must show that $B_\epsilon(y) \cap A \neq \emptyset$.

Since $\sum\limits_{i=1}^{\infty} y_i^2$ converges, we can pick an N such that $\sum\limits_{i=N+1}^{\infty} y_i^2 < \frac{\epsilon^2}{2}$. Using this value of N, choose rational numbers a_i so that $|a_i - y_i| < \frac{\epsilon}{\sqrt{2N}}$ for $i = 1, ..., N$. Define $a = (a_1, ..., a_N, 0, 0, 0, ...)$. Then $a \in A$ and

$$d(a, y) = \sqrt{\sum_{i=1}^{\infty}(a_i - y_i)^2} = \sqrt{\sum_{i=1}^{N}(a_i - y_i)^2 + \sum_{i=N+1}^{\infty}(a_i - y_i)^2}$$

$$= \sqrt{\sum_{i=1}^{N}(a_i - y_i)^2 + \sum_{i=N+1}^{\infty} y_i^2} < \sqrt{N \cdot \frac{\epsilon^2}{2N} + \frac{\epsilon^2}{2}} = \sqrt{\epsilon^2} = \epsilon.$$

Therefore $a \in B_\epsilon(y) \cap A$.

<u>We also claim</u> that int $A = \emptyset$. To prove this, we need to show that if $x = (x_i) \in A$, then no ball centered at x is a subset of A. To see this, pick <u>any</u> $\epsilon > 0$ and choose an irrational y_1 such that $|y_1 - x_1| < \epsilon$. Create a new point y from x by changing x_1 to y_1 so that $y = (y_1, x_2, ..., x_n, ...)$. Clearly, $y \in \ell_2$, $d(x, y) = |x_1 - y_1| < \epsilon$ and $y \notin A$, so $B_\epsilon(x) \not\subseteq A$.

What is Fr A ?

5) In any pseudometric space (X, d), $B_\epsilon(a)$ is a subset of the closed set $\{x \in X : d(a, x) \leq \epsilon\}$. Therefore cl $B_\epsilon(a) \subseteq \{x \in X : d(a, x) \leq \epsilon\}$.

But these two sets are <u>not necessarily equal</u>: <u>sometimes</u> the closed ball is larger than the closure of the open ball $B_\epsilon(a)$! For example, suppose d is the discrete unit metric on a set X where $|X| > 1$. Then

$$\{a\} = B_1(a) = \text{cl}\, B_1(a) \underset{\neq}{\subseteq} X = \{x \in X : d(a, x) \leq 1\}$$

Definition 4.7 Let (X, d) be a pseudometric space and $D \subseteq X$. We say that D is <u>dense</u> in (X, d) if cl $D = X$. The space (X, d) is called <u>separable</u> if it possible to find a countable dense set D in X. (*Note the spelling: "sep<u>a</u>rable," <u>not</u> "sep<u>e</u>rable."*)

Notice that "separable" is defined in terms of closure "cl" and the closure operator depends only on the topology, not the particular metric that generates the topology. Therefore separability is a topological property: if (X, d) is separable and $d' \sim d$, then (X, d') is also separable.

More informally, "D is dense in X" means that each point $x \in X$ can be approximated arbitrarily closely by a point from D. If X is uncountable, then a countable dense set D in X (if one exists) is a relatively small set which we can use to approximate any point in X arbitrarily closely.

Example 4.8

1) $\mathbb{R}^n$ is separable because $\mathbb{Q}^n$ is a countable dense set in $\mathbb{R}^n$; in particular, $\mathbb{Q}$ is a countable dense set in $\mathbb{R}$, so $\mathbb{R}$ is separable. $\mathbb{P}$ is an example of an <u>uncountable</u> dense subset of $\mathbb{R}$.

2) If X is countable, then (X, d) is automatically separable, because X is dense in X.

3) Suppose $\mathcal{T}_d$ is the discrete topology on X. Then every subset of X is closed, so a subset D is dense iff $D = X$. Therefore (X, d) is separable iff X is countable.

Suppose $\mathcal{T}_d$ is the trivial topology on X. Then $\operatorname{cl} D = X$ for <u>every</u> nonempty subset D since the only closed set containing D is X. So every nonempty subset D is dense and therefore (X, d) is separable – because, for example, any one-point set $\{x\}$ is dense.

4) The set $A = \mathbb{Q}^{\mathbb{N}} \cap \ell_2$ is dense in ℓ_2 (*see Example 4.6(4)*). This set A is <u>uncountable</u> because every sequence of rationals (x_i) with $|x_i| \leq \frac{1}{i}$ is in A (*why?*), and there are c such sequences (*why?*). However, (ℓ_2, d) <u>is</u> separable. Can you find a <u>countable</u> dense set D? (*The computation in Example 4.6.4 might give you an idea.*)

Definition 4.9 Let (X, d) be a pseudometric space and suppose A, B are nonempty subsets of X. We define the distance between A and B by

$$\operatorname{dist}(A, B) = \inf\{d(a, b) : a \in A \text{ and } b \in B\}.$$

We usually abbreviate $\operatorname{dist}(A, B)$ by $d(A, B)$ even though this is an "abuse of notation."

If $A \cap B \neq \emptyset$, then clearly $d(A, B) = 0$. But <u>notice</u>: if $A \cap B = \emptyset$, you <u>cannot</u> conclude that $d(A, B) > 0$, <u>not even</u> if A and B are both closed sets. For example, let A be the y-axis in $\mathbb{R}^2$ and $B = \{(x, y) \in \mathbb{R}^2 : y = \frac{1}{x}\}$. Then A and B are disjoint, <u>closed</u> sets but $d(A, B) = 0$.

We also go one step further and abbreviate $d(\{a\}, B)$ as $d(a, B) =$ "the distance from a to set B."

If $a \in B$, then clearly $d(a, B) = 0$. But the converse may not be true. For example, let $B = \{1, \frac{1}{2}, ..., \frac{1}{n}, ...\} \subseteq \mathbb{R}$. Then $d(0, B) = 0$ even though $0 \notin B$.

We can use "distance from a point to a set" to describe the closure of a set.

Theorem 4.10 Suppose (X, d) is a pseudometric space and that $A \subseteq X$. Then $x \in \operatorname{cl} A$ iff $d(x, A) = 0$.

Proof $x \in \operatorname{cl} A$ iff for every $\epsilon > 0$, $B_\epsilon(x) \cap A \neq \emptyset$
iff for every $\epsilon > 0$ there is a $y \in A$ with $d(x, y) < \epsilon$
iff $d(x, A) = 0$. •

Exercises

E10. Let (X, d) be a pseudometric space. Prove or disprove each statement.

a) $B_\epsilon(x)$ is never a closed set.
b) If $A \subseteq X$, then $\text{Fr}\, A = \text{cl}\, A - \text{int}\, A$.
c) For any $A \subseteq X$, $\text{diam}\,(A) = \text{diam}\,(\text{cl}\, A)$.
(*The diameter of a set A in a metric space is defined to be sup* $\{d(x, y) : x, y \in A\} \leq \infty$.)
d) For any $A \subseteq X$, $\text{diam}\,(A) = \text{diam}\,(\text{int}\, A)$.
e) For any $A \subseteq X$, $\text{int}(A \cup B) = \text{int}\, A \cup \text{int}\, B$.
f) For every $x \in X$ and $\epsilon > 0$, cl $(B_\epsilon(x))$ = $\{y \in X : \ d(x, y) \leq \epsilon\}$.

E11. a) Give an example of a metric space (X, d) that contains a proper nonempty clopen subset.
b) Give an example of a metric space (X, d) and a subset that is neither open nor closed.
c) Give an example of a metric space (X, d) and a subset A for which every point in A is a limit point of A. (*Note: a point x is called a* <u>*limit point of a set*</u> *A if, for every open set O containing x, $O \cap (A - \{x\}) \neq \emptyset$.*)
d) Give an example of a metric space (X, d) and a nonempty subset A such that every point is a limit point of A but $\text{int}(A) = \emptyset$. Can you also arrange that A is closed in X?
e) For each of the following subsets of $\mathbb{R}$, find the interior, closure and frontier ("boundary") in $\mathbb{R}$. Which points of the set are isolated in $\mathbb{R}$? which points of the set are isolated in the set ?

i) $A = \{m + n\pi : m, n \in \mathbb{N}\}$
ii) $B = \{\frac{1}{m} + \frac{1}{n} : m, n \in \mathbb{Z}\}$

E12. An infinite union of closed sets need not be closed. However if infinitely many closed sets are "spread out" enough from each other, then their union is closed. Parts a) and b) illustrate this.

a) Suppose that for each $n \in \mathbb{N}$, F_n is a closed set in $\mathbb{R}$ and that $F_n \subseteq (n, n+1)$. Prove that $\bigcup_{n=1}^{\infty} F_n$ is closed in $\mathbb{R}$.

b) More generally, suppose F_α is a closed set in (X, d) for each α in some index set A; suppose also that for each point $x \in X$ there is an $\epsilon > 0$ such that $B_\epsilon(x) \cap F_\alpha = \emptyset$ <u>for all but at most finitely many</u> α's.
Prove that $\bigcup_{\alpha \in A} F_\alpha$ is closed in (X, d). (*Notice that b) $\Rightarrow$ a). Why?*)

E13. a) Give an example of 2^c subsets of $\mathbb{R}$ all of which have the same closure. Do the same in $\mathbb{R}^2$.

b) Prove or disprove: there exist 2^c subsets of $\mathbb{R}$ such that any two have different closures. Is the situation the same in $\mathbb{R}^2$?

E14. The <u>Hilbert cube</u>, H, is a subset of ℓ_2: $H = \{x \in \ell_2 : \ |x_i| \leq \frac{1}{i}\}$. Prove that H is closed in ℓ_2. Prove or disprove: H is open in ℓ_2.

E15. In (X, d), a subset A is called a G_δ set if A can be written as a countable intersection of open sets; and A is called an F_σ set if A can be written as a countable union of closed sets.

Note: Open sets are often denoted using letters like O, U, or V (from open, and from the French ouvert), and sometimes by the letter G —from older literature where the German word was "Gebiet." Closed sets often are denoted by the letter F—from the French "ferme." Preferring these particular letters, of course, is just a common tradition – but many topologists follow it and most would wince to read something like "let F be an open set."

The names G_δ and F_σ go back to the classic book Mengenlehre by the German mathematician Felix Hausdorff. The σ and the δ in the notation come from the German words used for union and intersection: Summe and Durchschnitt.

a) Prove that every closed set in a pseudometric space (X, d) is a G_δ set, and every open set is an F_σ set.

b) Prove that the set of irrationals, $\mathbb{P}$, is a G_δ set in $\mathbb{R}$.

c) Find the error in the following argument which "proves" that every subset of $\mathbb{R}$ is a G_δ set:

Let $A \subseteq \mathbb{R}$. For $x \in A$, let $J_n = \bigcup\{B_{\frac{1}{n}}(x) : x \in A\}$. J_n is open for each $n \in \mathbb{N}$. Since $\{x\} = \bigcap_{n=1}^{\infty} B_{\frac{1}{n}}(x)$, it follows that $A = \bigcap_{n=1}^{\infty} J_n$, so A is a countable intersection of open sets, that is, A is a G_δ set.

c) In part b), the truth is that $\bigcap_{n=1}^{\infty} J_n = ?$

d) Suppose we list the members of $\mathbb{Q}$: $x_1, x_2, ..., x_n, ...$. Let $J = \bigcup_{n=1}^{\infty} B_{\frac{1}{n}}(x_n)$ where, of course, $B_{\frac{1}{n}}(x_n)$ is the interval $(x_n - \frac{1}{n}, x_n + \frac{1}{n}) \subseteq \mathbb{R}$. Is $J = \mathbb{R}$?

E16. Let (X, d) be a pseudometric space. Suppose that for every $\epsilon > 0$, there exists a countable subset D_ϵ of X with the following property: $\forall x \in X,\ \exists y \in D_\epsilon$ such that $d(x, y) < \epsilon$. Prove that (X, d) is separable.

E17. Suppose that X is an uncountable set and d is any metric on X which produces the discrete topology. (*Such a metric d might not be a constant multiple of the discrete unit metric: compare Example 2.14.2*). Show that for some $\epsilon > 0$ there is an uncountable subset A of X such that $d(x, y) \geq \epsilon$ for all $x \neq y \in A$.

E18. Let (X, d) be an infinite metric space. Prove that there exists an open set U such that both U and $X - U$ are infinite. *Hint: Consider a non-isolated point, if one exists.*

E19. A metric space (X, d) is called <u>extremally</u> <u>disconnected</u> if the closure of every open set is open. (*Note: "extremally" is the correct spelling; it is not the same as the everyday word "extremely."*)

Prove that if (X, d) is extremally disconnected, then the topology $\mathcal{T}_d$ is the discrete topology.

E20. In Definition 4.9, the function "dist" provides a measure of "distance" between <u>nonempty</u> subsets of (X, d). Is $(\mathcal{P}(X) - \{\emptyset\}, \text{dist})$ a metric (or pseudometric) space?

E21. Suppose (x_n) is a sequence in (X, d). We say that x_0 is a <u>cluster point</u> of (x_n) if for every open set O containing x_0 and for all $n \in \mathbb{N}$, $\exists k > n$ such that $x_k \in O$. (*This is clearly equivalent to saying that* $\forall \epsilon > 0$ *and* $\forall n \in \mathbb{N}$, $\exists k > n$ *such that* $x_k \in B_\epsilon(x_0)$.) Informally, x_0 is a cluster point of (x_n) if the sequence is "frequently in every open set containing x_0."

a) Show that there is a sequence in $\mathbb{R}$ for which every real number r is a cluster point.

b) A neurotic mathematician is walking along $\mathbb{R}$ from 0 toward 1. Halfway to 1, she remembers that she forgot something at 0 and starts back. Halfway back to 0, she decides to go to 1 anyway and turns around, only to change her mind again after traveling half the remaining distance to 1. She continues in this back-and-forth fashion forever. Find the cluster point(s) of the sequence (x_n), where x_n is the point where she reverses direction for the nth time.

5. Continuity

Suppose $a \in A \subseteq \mathbb{R}$ and that $f : A \to \mathbb{R}$. In elementary calculus, the set A is usually an interval, and the idea of continuity at a point a in A is introduced very informally. Roughly, it means that "if x is a point in the domain and near a, then $f(x)$ is near $f(a)$." In advanced calculus or analysis, the idea of "continuity of f at a" is defined carefully. The intuitive version of continuity – stated in terms of "nearness" – is made precise by measuring distances:

f is continuous at a means:

$$\forall\, \epsilon > 0 \;\; \exists\, \delta > 0 \text{ such that if } x \in A \text{ and } |x - a| < \delta, \text{ then } |f(x) - f(a)| < \epsilon.$$

The important thing to notice here is that we use the distance function in $\mathbb{R}$ to write the definition: $|x - a| = d(x, a)$ and $|f(x) - f(a)| = d(f(x), f(a))$. Since we have a way to measure distances in pseudometric spaces, we can make an a completely analogous definition of continuity for functions from one pseudometric space to another.

Definition 5.1 Let $f : X \to Y$, where (X, d) and (Y, s) are pseudometric spaces, and $a \in X$. We say that <u>f is</u> <u>continuous at a</u> if:

$$\forall \epsilon > 0 \;\; \exists \delta > 0 \text{ such that if } x \in X \text{ and } d(a, x) < \delta, \text{ then } s(f(x), f(a)) < \epsilon.$$

Notice that the sets X and Y may have completely unrelated metrics d and s: d measures distances in X and s measures distances in Y. But the idea is exactly the same as in calculus: "continuity of f at a" means, roughly, that "if x is near a in the domain X, then the image $f(x)$ is near $f(a)$ in Y."

Theorem 5.2 Suppose (X, d) and (Y, s) are pseudometric spaces. If $a \in X$ and $f : X \to Y$, then the following statements are equivalent:

1) f is continuous at a
2) $\forall \epsilon > 0\ \exists \delta > 0$ such that $f\,[B_\delta\,(a)] \subseteq B_\epsilon(f(a))$
3) $\forall \epsilon > 0\ \exists \delta > 0\ B_\delta\,(a) \subseteq f^{-1}[B_\epsilon\,(f(a))]$
4) $\forall N \subseteq Y$: if $f\,(a) \in \operatorname{int} N$, then $a \in \operatorname{int}\ f^{-1}[N]$.

Proof It is clear that conditions 1) - 2) - 3) are just equivalent restatements the definition of continuity at a in terms of images and inverse images of balls. Condition 4), however, seems a bit strange. We will show that 3) and 4) are equivalent.

3) $\Rightarrow$ 4) Suppose $f(a) \in \operatorname{int} N$. By definition of interior, there is an $\epsilon > 0$ such that $B_\epsilon(f(a)) \subseteq N$, so that $f^{-1}[B_\epsilon(f(a))] \subseteq f^{-1}[N]$. By 3), we can pick $\delta > 0$ so that $a \in B_\delta\,(a) \subseteq f^{-1}[B_\epsilon\,(f(a))] \subseteq f^{-1}[N]$. Since the δ-ball at a is an open subset of $f^{-1}[N]$, we get that $a \in \operatorname{int} f^{-1}[N]$, as desired.

4) $\Rightarrow$ 3) Suppose $\epsilon > 0$ is given. Let $N = B_\epsilon(f(a))$. Then N is open and $f(a) \in N = \operatorname{int} N$. We conclude from 4) that $a \in \operatorname{int} f^{-1}[N] = \operatorname{int} f^{-1}[B_\epsilon(f(a))]$. Therefore for some $\delta > 0$, $B_\delta\,(a) \subseteq \operatorname{int} f^{-1}[B_\epsilon(f(a))] \subseteq f^{-1}[B_\epsilon(f(a))]$, so 3) holds. •

Definition 5.3 If $N \subseteq X$ and $x \in \operatorname{int} N$, then N is called a <u>neighborhood of x</u> in (X, d). Thus, N is a neighborhood of x if there is an open set O such that $x \in O \subseteq N$.

Notice that:

1) The term "neighborhood" goes together with a point $x \in X$. We might say "O is an open set in X," but we would never say "N is a neighborhood in X" – but rather "N is a neighborhood <u>of x</u> in X," where x is some point in int(N).

2) <u>A neighborhood N of x need not be an open set</u>. However, be aware that in some books "a neighborhood of x" means "an open set containing x." It's not really important which way we make the definition of neighborhood (each version has its own technical advantages), but it is important that we all agree in these notes.

So in $\mathbb{R}^n$, for example, we say that the closed ball $N = \{(a, x) : d(a, x) \leq \epsilon\}$ is a neighborhood of a; in fact N is a neighborhood of each point x in its interior. A point x for which $d(a, x) = \epsilon$ is <u>in</u> N, but N is not a neighborhood of such a point x.

The following observation is almost trivial but it is important enough to state and remember.

Theorem 5.4 A subset N in (X, d) is open iff N is a neighborhood of each of its points.

Proof Suppose N is open. Then $\operatorname{int} N = N$, so each point x of N is automatically in $\operatorname{int} N$. So N is a neighborhood of each of its points.

Conversely, if N is a neighborhood of each of its points, then for every $x \in N$, we have $x \in \operatorname{int} N$. Therefore $N \subseteq \operatorname{int} N$, so $N = \operatorname{int} N$ and N is open. •

With this new terminology, we can restate the equivalence of 1) and 4) in Theorem 5.2 as:

f is continuous at a iff

whenever N is a neighborhood of $f(a)$ (*in* Y), then $f^{-1}[N]$ is a neighborhood of a (*in* X).

<u>This tells us something very important</u>. Interiors (and therefore neighborhoods of points) are defined in terms of the open sets in X – without needing to mention the distance function. This means that the neighborhoods of a point x depend only on the topology, not on the specific metric that generates the topology. Therefore <u>whether or not f is continuous at a does not actually depend on the specific metrics but only on the topologies in the domain and range</u>. In other words, "f is continuous at a" is a topological property.

For example, the function $\sin : \mathbb{R} \to \mathbb{R}$ is continuous at each point a in $\mathbb{R}$, and this is remains true if we measure distances in the domain with, say, the taxicab metric d_t and distances in the range with the max-metric d^* – since these are both equivalent to the usual metric d on $\mathbb{R}$.

We now define "f is a continuous function" in the usual way.

Definition 5.5 Suppose (X, d) and (Y, s) are pseudometric spaces. We say that $f : X \to Y$ is <u>continuous</u> if f is continuous at each point of X.

Theorem 5.6 Suppose $f : X \to Y$ where (X, d) and (Y, s) are pseudometric spaces. The following are equivalent:

1) f is continuous
2) if O is open in Y, then $f^{-1}[O]$ is open in X
3) if F is closed in Y, then $f^{-1}[F]$ is closed in X.

Proof 1) $\Rightarrow$ 2) Suppose O is open in Y and that $x \in f^{-1}[O]$. Since O is a neighborhood of $f(x)$ and f is continuous at x, we know from Theorem 5.2 that $f^{-1}[O]$ is a neighborhood of x. Therefore $f^{-1}[O]$ is a neighborhood of each of its points, so $f^{-1}[O]$ is open.

2) $\Rightarrow$ 3) If F is closed in Y, then $Y - F$ is open. By 2), $f^{-1}[Y - F] = X - f^{-1}[F]$ is open in X, so $X - (X - f^{-1}[F]) = f^{-1}[F]$ is closed in X.

3) $\Rightarrow$ 2) Exercise

2) $\Rightarrow$ 1) Suppose $a \in X$ and that N is a neighborhood of $f(a)$ in Y, so that $f(a) \in \text{int}\, N \subseteq N$. By 2), $f^{-1}[\text{int}\, N]$ is open in X, and $a \in f^{-1}[\text{int}\, N] \subseteq f^{-1}[N]$. Therefore $f^{-1}[N]$ is a neighborhood of a. Therefore f is continuous at a. Since a was an arbitrary point in X, f is continuous. •

Notice again: Theorem 5.6 shows that continuity is completely described in terms of the open sets (or equivalently, the closed sets), and the proof of the theorem is phrased entirely in terms of open (closed) sets, without any explicit mention of the pseudometrics on X and Y. Replacing d and s with equivalent pseudometrics would not affect the continuity of f.

Theorem 5.7 Suppose (X, d), (Y, s) and (Z, t) are pseudometric spaces and that $f : X \to Y$ and $g : Y \to Z$. If f is continuous at $a \in X$ and g is continuous at $f(a) \in Y$, then $g \circ f$ is continuous at a. (Therefore, if f and g are continuous, so is $g \circ f$.)

Proof If N is a neighborhood of $g(f(a))$, then $g^{-1}[N]$ is a neighborhood of $f(a)$, because g is continuous at $f(a)$. Since f is continuous at a, $f^{-1}[g^{-1}[N]]$ is a neighborhood of a. But $f^{-1}[g^{-1}[N]] = (g \circ f)^{-1}[N[$, so $g \circ f$ is continuous at a. •

Example 5.8

1) Suppose $f : \mathbb{R} \to \mathbb{R}$ is given by $f(x) = \begin{cases} 1, & \text{if } x = 0 \\ 0, & \text{if } x \neq 0 \end{cases}$. Then $N = (\frac{1}{2}, \frac{3}{2})$ is a neighborhood of $f(0) = 1$, but $f^{-1}[N] = \{0\}$ is not a neighborhood of 0. Therefore f is not continuous at 0. To see the same thing using slightly different language: f is not continuous at 0 because, choosing $\epsilon = \frac{1}{2}$, there is no choice of $\delta > 0$ such that $f[B_\delta(0)] \subseteq B_\epsilon(f(0)) = B_\epsilon(1)$.

2) If $f : (X, d) \to (Y, s)$ is a constant function, then f is continuous. To see this, suppose $f(x) = c$ for every x. If O is open in Y, then $f^{-1}[O] = \begin{cases} \emptyset & \text{if } c \notin O \\ X & \text{if } c \in O \end{cases}$. In both cases, $f^{-1}[O]$ is open.

3) Suppose $f : \mathbb{R} \to \mathbb{R}$ is given by $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{P} \end{cases}$. Then f is not continuous at any point $a \in \mathbb{R}$ (why?). However $f|\mathbb{Q} = g : \mathbb{Q} \to \mathbb{R}$ is continuous at every point of $\mathbb{Q}$, because g is a constant function.

There is a curious old result called <u>Blumberg's Theorem</u> which states:

For any $f : \mathbb{R} \to \mathbb{R}$, there exists a dense set $D \subseteq \mathbb{R}$ such that $f|D = g : D \to \mathbb{R}$ is continuous.

Blumberg's Theorem is rather difficult to prove, and not very useful.

4) Suppose $f, g : X \to \mathbb{R}$ where (X, d) is a pseudometric space. Since these functions are real-valued, it makes sense to define functions $f + g$, $f - g$, $f \cdot g$ and $\frac{f}{g}$ in the obvious way. For example, $(f + g)(x) = f(x) + g(x)$, where the " $+$ " on the right is ordinary addition in $\mathbb{R}$.

If f and g are continuous at a point $a \in X$, then the functions $f + g$, $f - g$, and $f \cdot g$ are also continuous at a; and $\frac{f}{g}$ is continuous at a if $g(a) \neq 0$. The proofs are just like those given in calculus (where $X = \mathbb{R}$)

For example, consider $f + g$: given $\epsilon > 0$, then (because f, g are continuous at a), we can find $\delta_1 > 0$ and $\delta_2 > 0$ so that

if $d(x, a) < \delta_1$, then $|f(x) - f(a)| < \frac{\epsilon}{2}$ and
if $d(x, a) < \delta_2$, then $|g(x) - g(a)| < \frac{\epsilon}{2}$

Let $\delta = \min\{\delta_1, \delta_2\}$. Then if $d(x, a) < \delta$, we have

$$|(f + g)(x) - (f + g)(a)| = |f(x) - f(a) + g(x) - g(a)|$$
$$\leq |f(x) - f(a)| + |g(x) - g(a)| < \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon$$

You can find at the other proofs in an analysis textbook.

5) For $x = (x_1, x_2, ..., x_n, ...) \in \ell_2$, define $f : \ell_2 \to \mathbb{R}$ by $f(x) = x_1$. (f *is a "projection" of* ℓ_2 *onto* $\mathbb{R}$.) Then f is continuous at x. To see this, suppose $\epsilon > 0$, and let $\delta = \epsilon$. If $y = (y_1, y_2, ..., y_n, ...) \in B_\delta(x)$, then

$$|f(x) - f(y)| = |x_1 - y_1| \leq d(x, y) = (\sum_{i=1}^{\infty}(x_i - y_i)^2)^{1/2} < \delta = \epsilon,$$

so $f[B_\delta(x)] \subseteq B_\epsilon(f(x))$.

A similar argument shows that each projection $g(x) = x_n$ is also continuous, and an argument only slightly more complicated would show, for example, that the projection function $h : \ell_2 \to \mathbb{R}^3$ given by $h(x) = (x_3, x_9, x_{11})$ is continuous.

6) If a is an isolated point in (X, d), then every function $f : (X, d) \to (Y, s)$ is continuous at a. To see this, suppose N is a neighborhood of $f(a)$. Then $a \in \{a\} \subseteq f^{-1}[N]$. But $\{a\}$ is open in X, so $f^{-1}[N]$ is a neighborhood of a.

If $\mathcal{T}_d$ happens to be the discrete topology, then <u>every</u> point in X is isolated, so f is continuous (*In this case, we could argue instead that whenever O is open in Y then $f^{-1}[O]$ must be open in X – because every subset of X is open.*)

7) A function $f : (X, d) \to (Y, s)$ is called an <u>isometry of X into Y</u> if it preserves distances, that is, if $d(a, b) = s(f(a), f(b))$ for all $a, b \in X$. An isometry is clearly continuous (given $\epsilon > 0$, choose $\delta = \epsilon$).

Note that if d is a <u>metric</u>, then f one-to-one (*Why?*). If f happens to be a bijection, we say that (X, d) and (Y, s) are <u>isometric</u> to each other. In that case, it is clear that the inverse function f^{-1} is also an isometry, so f^{-1} is also continuous.

Theorem 5.9 If (X, d) is a <u>metric</u> space and $x \neq y \in X$, then there exist open sets U and V such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$. (*More informally: distinct points in a metric space can be separated by disjoint open sets.*)

Proof Since $x \neq y$, $d(x, y) = \delta > 0$. Let $U = B_{\frac{\delta}{2}}(x)$ and $V = B_{\frac{\delta}{2}}(y)$. These sets are open, and if there were a point $z \in U \cap V$, we would have a contradiction :

$$\delta = d(x, y) \leq d(x, z) + d(z, y) < \tfrac{\delta}{2} + \tfrac{\delta}{2} = \delta. \quad \bullet$$

Theorem 5.9 may not be true if d is not a metric. For example, if d is the trivial pseudometric on X, then the only open sets containing x and y are $U = V = X$.)

Example 5.10

1) Suppose $f : (X, d) \to (Y, s)$, where d is the trivial pseudometric on X and Y is any <u>metric</u> space. We already know that if f is constant, then f is continuous.

If f is not constant, then there are points $a, b \in X$ for which $f(a) \neq f(b)$. Since s is a metric, we can pick disjoint open sets U and V in Y with $f(a) \in U$ and $f(b) \in V$. Then $f^{-1}[U] \neq \begin{cases} \emptyset & \text{(because } a \in f^{-1}[U]) \\ X & \text{(because } b \notin f^{-1}[U]) \end{cases}$

Since $\mathcal{T}_d = \{\emptyset, X\}$, we see that $f^{-1}[U]$ is not open so f is not continuous.

So, in this situation we conclude that: f is continuous <u>iff</u> f is constant.

2) Suppose d is the usual metric and s is the discrete unit metric on $\mathbb{R}$. Let $i : (\mathbb{R}, d) \to (\mathbb{R}, s)$ is the identity map $i(x) = x$. For every open set O in $(\mathbb{R}, d)$, the image set $i[O]$ is open in $(\mathbb{R}, s)$, but this is not the criterion for continuity: in fact, this function is <u>not continuous at any point</u>. The criterion for continuity is that the <u>inverse</u> image of every open set must be open.

Example 5.10.2 leads us to a definition.

Definition 5.11 A function $f : (X, d) \to (Y, s)$ is called an <u>open function</u> or <u>open mapping</u> if: whenever O is open in X, then the image set $f[O]$ is open in Y. Similarly, we call f a <u>closed mapping</u> if whenever F is closed in X, then the image set $f[F]$ is closed in Y.

The identity mapping i in Example 5.10.2 is <u>both</u> open and closed but i is not continuous. You can convince yourself fairly easily that the projection function $\pi_x : \mathbb{R}^2 \to \mathbb{R}$ given by $\pi_x(x, y) = x$ is open and continuous, but it is not closed – for example, the set $F = \{(x, y) : y = \frac{1}{x}\}$ is a closed set in $\mathbb{R}^2$ but $\pi_x[F] = (0, \infty)$ is not closed in $\mathbb{R}$. The general point is that <u>for a function</u> $f : (X, d) \to (Y, s)$, <u>the properties "open", "closed", and "continuous" are completely independent.</u> You should provide other examples: for instance, a function that is continuous but not open or closed.

With just the basic ideas about continuous functions, we can already prove some rather interesting results.

Theorem 5.12 Suppose $f, g : (X, d) \to (Y, s)$, where d is a pseudometric and s is a <u>metric</u>. Let D be a dense subset of X. If f and g are continuous and $f|D = g|D$, then $f = g$. (*More informally: if two continuous functions with values in a metric space agree on a dense set, then they agree everywhere.*)

Proof Suppose $f \neq g$. Then, for some point $a \in X$, we have $f(a) \neq g(a)$. Since s is a metric, we can find <u>disjoint</u> open sets U and V in Y with $f(a) \in U$ and $g(a) \in V$. Since f and g are continuous at a, there are open sets U_1 and V_1 in X that contain a and that satisfy $f[U_1] \subseteq U$ and $g[V_1] \subseteq V$.

$U_1 \cap V_1$ is an open set containing a. Since $a \in \operatorname{cl} D$, there must be a point $d \in (U_1 \cap V_1) \cap D$. Then $f(d) \neq g(d)$ because $f(d) \in U$, $g(d) \in V$ and $U \cap V = \emptyset$. Therefore $f|D \neq g|D$. •

Example 5.13 If $f, g \in C(\mathbb{R})$ and $f|\mathbb{Q} = g|\mathbb{Q}$, then $f = g$ by Theorem 5.12. In other words, the mapping $\Phi : C(\mathbb{R}) \to C(\mathbb{Q})$ given by $\Phi(f) = f|\mathbb{Q} \in C(\mathbb{Q})$ is one-to-one. Therefore $|C(\mathbb{R})| \leq |C(\mathbb{Q})| \leq c^{\aleph_0} = c$.

On the other hand, each <u>constant</u> function $f(x) = r$ is in $C(\mathbb{R})$, so $|C(\mathbb{R})| \geq c$. It follows that $|C(\mathbb{R})| = c$. In other words, <u>there are exactly c continuous functions from $\mathbb{R}$ to $\mathbb{R}$</u>.

Example 5.14 Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy the functional equation

$$f(x + y) = f(x) + f(y) \text{ for all } x, y \in \mathbb{R} \qquad (*)$$

Simple induction shows that for $n \in \mathbb{N}$, $f(x_1 + ... + x_n) = f(x_1) + ... + f(x_n)$.

By $(*)$,

$$f(0) = f(0 + 0) = f(0) + f(0), \text{ so } f(0) = 0.$$

Let $f(1) = c$. Then

$$\begin{aligned} f(2) &= f(1 + 1) = f(1) + f(1) = c \cdot 2 \\ f(3) &= f(2 + 1) = f(2) + f(1) = c \cdot 2 + c = c \cdot 3 \\ &\vdots \end{aligned}$$

Continuing, we see that $f(n) = cn$ for every $x \in \mathbb{N}$. Similarly, for each $m, n \in \mathbb{N}$, we have

$$f(1) = f(\underbrace{\tfrac{1}{n} + ... + \tfrac{1}{n}}_{n \text{ terms}}) = f(\tfrac{1}{n}) + ...f(\tfrac{1}{n}) = nf(\tfrac{1}{n}), \text{ so } f(\tfrac{1}{n}) = c(\tfrac{1}{n})$$

$$f(\tfrac{m}{n}) = f(\underbrace{\tfrac{1}{n} + \ldots + \tfrac{1}{n}}_{m \text{ terms}}) = mf(\tfrac{1}{n}) = m \cdot c(\tfrac{1}{n}) = c(\tfrac{m}{n})$$

So far, we have shown that a function f satisfying $(*)$ must have the formula $f(x) = cx$ for every positive rational $x = \frac{m}{n}$.

Since $\quad 0 = f(0) = f(\frac{m}{n} + (-\frac{m}{n})) = f(-\frac{m}{n}) + f(\frac{m}{n})$, we get that

$$f(-\tfrac{m}{n}) = -f(\tfrac{m}{n}) = -(\tfrac{m}{n})c = c(-\tfrac{m}{n}).$$

Therefore, $f(x) = cx$ for every $x \in \mathbb{Q}$.

So far, we have not used the hypothesis that f is continuous. Let $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = cx$. Since f and g are continuous and $f|\mathbb{Q} = g|\mathbb{Q}$, Theorem 5.12 tells us that $f = g$, that is, $f(x) = cx$ for <u>all</u> $x \in \mathbb{R}$.

> *The continuous functions satisfying the functional equation $*$) were first described by Cauchy in 1821. It turns out that there are also discontinuous functions f satisfying $(*)$, but they are not easy to find. In fact, they must satisfy a nasty condition called "nonmeasurability" (which makes them "extremely discontinuous").*

In calculus, another important is the idea of a convergent sequence (in $\mathbb{R}$ or $\mathbb{R}^n$). We can generalize the idea of a convergent sequence in $\mathbb{R}^n$ to any pseudometric space.

Definition 5.15 A <u>sequence</u> (x_n) <u>in (X, d)</u> <u>converges to x</u> $\in X$ if any one of the following (clearly equivalent) conditions holds:

1) $\forall \epsilon > 0 \ \exists N \in \mathbb{N}$ such that if $n \geq N$, then $x_n \in B_\epsilon(x)$
(that is, if the sequence of numbers $(d(x_n, x)) \to 0$ in $\mathbb{R}$)

2) if O is open and $x \in O \subseteq X$, then $\exists N \in \mathbb{N}$ such that if $n \geq N$, then $x_n \in O$

3) if W is a neighborhood of x, then $\exists N \in \mathbb{N}$ such that if $n \geq N$, then $x_n \in W$.

If (x_n) converges to X, we write $(x_n) \to x$.

Conditions 2) and 3) describe the convergence of sequences in terms of open sets (or neighborhoods) rather than directly using the distance functions. Therefore replacing d with an equivalent metric d' does not affect which sequences converge to which points: sequential convergence is a topological property.

If a sequence (x_n) has a certain property P for all $n \geq$ some N, we say that "(x_n) <u>eventually</u> has property P." For example, the sequence (0,3,1,7,7,7,7,...) is <u>eventually constant</u>; the sequence $(-1, -3, 5, -2, 5, 6, 7, 8, ..., n, n+1, ...)$ is <u>eventually increasing</u>. Using this terminology, we can give a completely precise definition of convergence by saying: (x_n) converges to x if (x_n) is eventually in each neighborhood of x.

Example 5.16 1) In $\mathbb{R}$, $(\frac{1}{n}) \to 0$.

2) Suppose d is the discrete unit metric on X. Then each set $\{x\}$ is open so $(x_n) \to x$ iff (x_n) is eventually in each neighborhood of X iff (x_n) is eventually in $\{x\}$. In other words, $(x_n) \to x$ iff $x_n = x$ eventually. Every convergent sequence is eventually constant.

At the other extreme, suppose d is the trivial pseudometric on X. Then <u>every</u> sequence (x_n) converges to <u>every</u> point $x \in X$ (since the only neighborhood of x is X).

Example 5.16.2 shows that limits of sequences in a pseudometric space do not have to be unique: the same sequence can have many limits. However if d is a <u>metric</u>, then sequential limits in (X, d) must be unique, as the following theorem shows.

Theorem 5.17 A sequence (x_n) in a <u>metric</u> space (X, d) has at most one limit.

Proof Suppose $x \neq y \in X$ and let U, V be disjoint open sets with $x \in U$ and $y \in V$. If $(x_n) \to x$, then (x_n) must be eventually in U. Since U and V are disjoint, this means that (x_n) cannot be eventually in V (*in fact, the sequence must be eventually outside* V), so $(x_n) \not\to y$. •

In a pseudometric space, sequences can be used to describe the closure of a set.

Theorem 5.18 Suppose $A \subseteq X$, where (X, d) is a pseudometric space. Then $x \in \operatorname{cl} A$ iff there is a sequence (a_n) in A for which $(a_n) \to x$.

Proof First, suppose there is a sequence (a_n) in A for which $(a_n) \to x$. If N is any neighborhood of x, then (a_n) is eventually in N. Therefore $N \cap A \neq \emptyset$, so $x \in \operatorname{cl} A$.

Conversely, suppose $x \in \operatorname{cl} A$. Then $B_{\frac{1}{n}}(x) \cap A \neq \emptyset$ for each $n \in \mathbb{N}$, so we can choose a point $a_n \in B_{\frac{1}{n}}(x) \cap A$. Then $(a_n) \to x$ (because $d(a_n, x) \to 0$). •

> *Note: the sequence (a_n) is actually a function $f : \mathbb{N} \to X$ with the property that $a_n = f(n) \in B_{\frac{1}{n}}(x) \cap A$. Informally, the existence of such a function is completely clear. But to be precise, this argument actually depends on the Axiom of Choice. The proof as written doesn't describe how to pick specific a_n's: it depends on making "arbitrary choices." But using AC gives us a function f which "chooses" one point from each set in the collection $\{B_{\frac{1}{n}}(x) \cap A : n \in \mathbb{N}\}$.*

Theorem 5.18 tells us something <u>very</u> <u>important</u> about the role of sequences in pseudometric spaces. The set A is closed iff $A = \operatorname{cl} A$. But $A \subseteq \operatorname{cl} A$ is always true, so we can say A is closed iff $\operatorname{cl} A \subseteq A$. But that is true iff the limits of convergent sequences (a_n) from A must also be in A. Therefore a complete knowledge about what sequences converge to what points in (X, d) would let you determine which sets are closed (and therefore, by taking complements, which sets are open). In other words, all the information about "which sets in (X, d) are open or closed?" is revealed by the convergent sequences. We summarize this by saying that <u>in a pseudometric space (X, d)</u>, <u>sequences are sufficient to describe the topology</u>.

Example 5.19 If d is a pseudometric on X, then d' defined by $d'(x,y) = \min\{1, d(x,y)\}$ is also a pseudometric on X. It is clear that $d'(x_n, x) \to 0$ iff $d(x_n, x) \to 0$. In other words, the metrics d and d' produce exactly the same convergent sequences and limits in (X, d). Since sequences are sufficient to determine the topology in pseudometric spaces, we conclude that d and d' are equivalent pseudometrics on X.

This example also shows that for any given (X, d) there is always an equivalent pseudometric d' on X for which all distances are ≤ 1: <u>every pseudometric is equivalent to a bounded pseudometric</u>.

Another modification of d that accomplishes the same thing is $d''(x,y) = \frac{d(x,y)}{1+d(x,y)}$. This time, it is a little harder to verify that d'' is in fact a pseudometric – the triangle inequality for d'' takes a bit of work. Clearly $d''(x,y) \leq 1$, and $d''(x_n, x) \to 0$ iff $d(x_n, x) \to 0$. So $d'' \sim d \sim d'$.

Definition 5.20 The <u>diameter</u> of a set A in (X, d) is defined by $\text{diam}(A) = \sup\{d(x,y) : x, y \in A\}$ (*we allow the possibility that diam*$(A) = \infty$). If A has finite diameter, we say that A is a <u>bounded set.</u>

It is an easy exercise to show that A is bounded iff $A \subseteq B_k(x_0)$ for some sufficiently large k (where x_0 can be any point in X).

The diameter of a set depends on the particular metric being used. Since we can always replace d by an equivalent metric d' or d'' for which $\text{diam}(X) \leq 1$, <u>boundedness is not a topological property</u>.

The fact that the convergent sequences determine the topology in (X, d) gives us an upper bound on the size of certain metric spaces.

Theorem 5.21 If D is a dense set in a <u>metric</u> space (X, d), then $|X| \leq |D|^{\aleph_0}$. In particular, for a <u>separable</u> metric space (X, d), it must be true that $|X| \leq \aleph_0^{\aleph_0} = c$.

Proof For each $x \in X$, pick a sequence (d_n) in D such that $(d_n) \to x$. This sequence is actually a function $f_x \in D^{\mathbb{N}}$. Since a sequence in a metric space has at most one limit, the mapping $\Phi : X \to D^{\mathbb{N}}$ given by $\Phi(x) = f_x$ is one-to-one, so $|X| \leq |D^{\mathbb{N}}| = |D|^{\aleph_0}$. •

> *Note: If $|D| = m > \aleph_0$, do not jump to the (possibly false) conclusion that $|X| \leq m^{\aleph_0} = m$. See the note of caution in Chapter I at the end of Example I.14.8.*

Theorem 5.21 is <u>not</u> true if (X, d) is merely a pseudometric space. For example, let X be an uncountable set (with arbitrarily large cardinality) and let d be the trivial pseudometric on X. Then any singleton $\{x\}$ is dense, but $|X| > 1^{\aleph_0} = 1$. In this case, where does the proof of Theorem 5.21 fall apart?

Sequences are sufficient to determine the topology in a pseudometric space, and continuity is characterized in terms of open sets, so it should not be a surprise that sequences can be used to decide whether or not a function f between pseudometric spaces is continuous.

Theorem 5.22 Suppose (X, d) and (Y, s) are pseudometric spaces, and that $f : X \to Y$. Then f is continuous at $a \in X$ iff $(f(x_n)) \to f(a)$ for <u>every</u> sequence $(x_n) \to a$.

Proof Suppose f is continuous at a and consider any sequence $(x_n) \to a$. If W is a neighborhood of $f(a)$, then $f^{-1}[W]$ is a neighborhood of a, so (x_n) is eventually in $f^{-1}[W]$. This implies that $(f(x_n))$ is eventually in W, so $(f(x_n)) \to f(a)$.

Conversely, if f is <u>not</u> continuous at a, then $\sim (\forall \epsilon > 0\ \exists \delta > 0\ \ f[B_\delta(x)] \subseteq B_\epsilon(f(a)))$, that is, $\exists \epsilon > 0\ \forall \delta > 0\ \ f[B_\delta(x)] \nsubseteq B_\epsilon(f(a))$.

For this ϵ and $\delta = \frac{1}{n}$, we have that $f[B_{\frac{1}{n}}(x)] \nsubseteq B_\epsilon(f(a))$. So for each n we can choose a point $x_n \in B_{\frac{1}{n}}(x)$ for which $f(x_n) \notin B_\epsilon(f(x))$. Then $(x_n) \to x$ (because $d(x_n, x) < \frac{1}{n} \to 0$), but $(f(x_n)) \not\to f(a)$ (because $s(f(x_n), f(a)) \geq \epsilon$ for all n). Therefore f is not continuous at a. •

Note: the <u>first half</u> of the proof is phrased completely in terms of neighborhoods of x_0 and $f(x_0)$; that part of the proof is topological. However the second half makes explicit use of the metric. In fact, as we shall see later, the second half of the proof <u>must</u> involve a little more than just the open sets.

Notice also that the second half of the proof makes uses the Axiom of Choice (the function x "chooses" x_n for each n.

The following theorem and its corollaries are often technically useful. Moreover, they show us that <u>a pseudometric space</u> (X, d) <u>has lots of "built-in" continuous functions</u> – these functions can be defined using pseudometric d.

Theorem 5.23 In a pseudometric space (X, d):

$$\text{if } (x_n) \to x \text{ and } (y_n) \to y, \text{ then } d(x_n, y_n) \to d(x, y).$$

Proof Given $\epsilon > 0$, pick N large enough so that $n \geq N$ implies that $d(x_n, x) < \frac{\epsilon}{2}$ and $d(y_n, y) < \frac{\epsilon}{2}$ are both true. Since $d(x_n, y_n) \leq d(x_n, x) + d(x, y_n) \leq d(x_n, x) + d(x, y) + d(y, y_n)$, we get that

$$\text{if } n \geq N, \quad d(x_n, y_n) < \tfrac{\epsilon}{2} + d(x, y) + \tfrac{\epsilon}{2} = d(x, y) + \epsilon \quad (*)$$

Similarly, $d(x, y) \leq d(x, x_n) + d(x_n, y_n) + d(y_n, y)$, so that

$$\text{if } n \geq N, \quad d(x, y) < \tfrac{\epsilon}{2} + d(x_n, y_n) + \tfrac{\epsilon}{2} = d(x_n, y_n) + \epsilon \ \ (**)$$

Combining $(*)$ and $(**)$ gives that $|d(x_n, y_n) - d(x, y)| < \epsilon$ if $n \geq N$, so $d(x_n, y_n) \to d(x, y)$. •

Corollary 5.24 Let (X, d) be a pseudometric space and $a \in X$. If $(x_n) \to x$ in X, then $d(x_n, a) \to d(x, a)$ (in $\mathbb{R}$).

Proof In Theorem 5.23, let (y_n) be the constant sequence where $y_n = a$ for every n. •

Corollary 5.25 Suppose $a \in (X, d)$. Define $f : X \to \mathbb{R}$ by $f(x) = d(x, a)$. Then f is continuous.

Proof Let x_0 be a point in X. If $(x_n) \to x_0$, then by Corollary 5.24, $(d(x_n, a)) \to d(x_0, a)$, that is $(f(x_n)) \to f(x_0)$. So f is continuous at x_0 (by Theorem 5.22). •

Recall that for a nonempty subset A of X, we defined $d(x, A) = \inf\{d(x, a) : a \in A\}$.
The following theorem is also useful.

Theorem 5.26 If A is a nonempty subset of the pseudometric space (X, d), then the function $f : X \to \mathbb{R}$ defined by $f(x) = d(x, A)$ is continuous.

Proof We show that f is continuous at each point $x_0 \in X$. Let $a \in A$. Then the following inequalities are true for every $x \in X$:

$$\begin{aligned} d(a, x_0) &\le d(a, x) + d(x, x_0) \\ d(a, x) &\le d(a, x_0) + d(x_0, x) \end{aligned}$$

Apply "$\inf_{a \in A}$" to each inequality to get

$$\begin{aligned} d(x_0, A) &\le d(x, A) + d(x, x_0) \text{ or} \qquad & d(A, x_0) - d(A, x) &\le d(x, x_0) \\ d(x, A) &\le d(x_0, A) + d(x_0, x) \text{ or} \qquad & d(A, x) - d(A, x_0) &\le d(x, x_0) \end{aligned}$$

so for all $x \in X$, $\quad |d(A, x) - d(A, x_0)| \le d(x, x_0)$. In other words,

for all $x \in X$, $\quad |f(x) - f(x_0)| \le d(x, x_0)$. $\quad$ (*)

So for $\epsilon > 0$, we can choose $\delta = \epsilon$. Then if $d(x, x_0) < \delta$, we have $|f(x) - f(x_0)| < \epsilon$. Therefore f is continuous at x_0. •

Comments on the proof:

i) For a given $\epsilon > 0$: the <u>same choice</u> $\delta = \epsilon$ can be used at every point x_0. A function f that satisfies this condition – stronger than mere continuity – is called <u>uniformly continuous</u>. We will say more about uniform continuity in Chapter IV.)

ii) From the last inequality (*) we could have argued instead: for any sequence $(x_n) \to x_0$, we have $d(x_n, x_0) \to 0$, and this forces $(f(x_n)) \to f(x_0)$. Therefore f is continuous at x_0.
However this argument obscures the observation about "uniform continuity" made in i).

Exercises

E22. Suppose $a \in A \subseteq \mathbb{R}$ and that $f : A \to \mathbb{R}$. We said that f is continuous at a if

$\forall\, \epsilon > 0\ \exists\, \delta > 0$ such that if $x \in A$ and $|x - a| < \delta$, then $|f(x) - f(a) < \epsilon$.

The <u>order of the quantifiers</u> is important. What functions are described by each of the following modifications of the definition?

a) $\forall\, \delta > 0\ \exists\, \epsilon > 0$ such that if $x \in A$ and $|x - a| < \delta$, then $|f(x) - f(a) < \epsilon$
b) $\forall\, \epsilon > 0\ \forall\, \delta > 0$ such that if $x \in A$ and $|x - a| < \delta$, then $|f(x) - f(a) < \epsilon$
c) $\exists\, \epsilon > 0\ \exists\, \delta > 0$ such that if $x \in A$ and $|x - a| < \delta$, then $|f(x) - f(a) < \epsilon$.

In each case, what happens if the restriction " > 0" is dropped on either ϵ or δ ?

E23. Suppose $f : \mathbb{R} \to \mathbb{R}$ and that $f(a) = b$. What does each of the following statements tell us about f? *<u>Throughout this exercise</u>, "interval" always means "a bounded open interval* (c, d).

a) For every interval I containing a and every interval J containing b, $f[I] \subseteq J$.

b) There exists an interval J containing b and there exists an interval I containing a such that $f[I] \subseteq J$.

c) There exists an interval J containing b such that for every interval I containing a, $f[I] \subseteq J$

d) There exists an interval J containing b such for every interval I containing a, $f[I] \not\subseteq J$

. e) For every interval I containing a, there exists an interval J containing b such that $f[I] \subseteq J$

f) There exists an interval I containing a such that for every interval J containing b, $f[I] \subseteq J$

E24. (<u>The Pasting Lemma, easy version</u>) *The two parts of the problem give conditions when a collection of continuous functions defined on subsets of X can be "united" (= "pasted together") to form a new continuous function.* Let sets O_α be open in (X, d), where $\alpha \in A$ (an indexing set of any size); and let $F_1, ..., F_n$ $(n \in \mathbb{N})$ be closed in (X, d).

a) If functions $f_\alpha : O_\alpha \to (Y, s)$ are continuous and $f_\alpha|(O_\alpha \cap O_\beta) = f_\beta|(O_\alpha \cap O_\beta)$ for all α and β in A *(that is, f_α and f_β agree where their domains overlap*), then prove that $\bigcup_{\alpha \in A} f_\alpha = f : \bigcup_{\alpha \in A} O_\alpha \to Y$ is continuous.

b) If functions $f_i : F_i \to (Y, s)$ are continuous for each $i = 1, ..., n$ and $f_i|(F_i \cap F_j) = f_j|(F_i \cap F_j)$ for all i and j *(that is, f_i and f_j agree where their domains overlap*), then prove that $f = \bigcup_{i=1}^n f_i : \bigcup_{i=1}^n F_i \to Y$ is continuous.

c) Give an example to show that b) may be false for an <u>infinite</u> collection of functions f_i $(i \in \mathbb{N})$ defined on closed subsets of X, <u>even if</u> the domains F_i are pairwise disjoint.

E25. A point x_0 in (X, d) is <u>a cluster point of the sequence</u> (x_n) if for every neighborhood N of x_0 and for all $n \in \mathbb{N}$, $\exists k > n$ such that $x_k \in N$. (*When this condition is true, we say that "(x_n) is frequently in every neighborhood of x_0."*)

Prove that if $f : (X, d) \to (Y, s)$ is continuous and x_0 is a cluster point of (x_n) in X, then $f(x_0)$ is a cluster point of the sequence $(f(x_n))$ in Y.

E26. The <u>characteristic function</u> of a set $A \subseteq X$ is defined by $\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$.
For which sets A is $\chi_A : \mathbb{R} \to \mathbb{R}$ continuous?

E27. Show that a set O is open in (X, d) if and only if there is a continuous function $f : X \to \mathbb{R}$ and an open set W in $\mathbb{R}$ such that $O = f^{-1}[W]$.

E28. Let $x \in X$, where (X, d) is a pesudometric space. Suppose (x_n) is a sequence in X such that $(f(x_n)) \to f(x)$ for <u>every</u> $f \in C(X)$. Either prove that $(x_n) \to x$, or give a example to show that this may be false.

E29. Let d be the usual metric on $\mathbb{R}$. If possible, find a metric d' on $\mathbb{R}$ such that $(x_n) \to 0$ with respect to d iff $(x_n) \to 0$ with respect to d', but d' is not equivalent to d.

E30. a) Suppose A is a closed set in the pseudometric space (X, d) and that $x_0 \notin A$. Prove that there is a continuous function $f : X \to [0, 1]$ such that $f|A = 0$ and $f(x_0) = 1$. (*Hint: Consider the function "distance to the set A."*)

b) Suppose A and B are disjoint closed sets in (X, d). Prove that there exists a continuous function $f : X \to \mathbb{R}$ such that $f|A = 0$ and $f|B = 1$. (*Hint: Consider* $\frac{d(x,A)}{d(x,A)+d(x,B)}$)

c) Using b) (*or by some other method*) prove that if A and B are disjoint closed sets in (X, d), then there exist open sets U and V such that $A \subseteq U$, $B \subseteq V$ and $U \cap V = \emptyset$. Can U and V always be chosen so that $\operatorname{cl} U \cap \operatorname{cl} V = \emptyset$?

E31. *A* function $f: (X, d) \to (Y, s)$ is called an <u>isometry of (X, d) into (Y, s)</u> if f preserves distances: that is, if $d(x, y) = s(f(x), f(y))$ for all $x, y \in X$. Such an f is also called an <u>isometric embedding</u>.

If f is also <u>onto</u>, we say that (X, d) and (Y, s) are <u>isometric</u> to each other. Otherwise, (X, d) is isometric to a subset of (Y, s)

Let $\mathbb{R}$ and $\mathbb{R}^2$ have their usual metrics.

a) Prove that $\mathbb{R}$ and $\mathbb{R}^2$ are not isometric to each other.

b) Let $a \in \mathbb{R}$. Prove that there are exactly two isometries from $\mathbb{R}$ onto $\mathbb{R}$ which hold the point a fixed (that is, for which $f(a) = a$).

c) Give an example of a metric space (X, d) which is isometric to a proper subset of itself.

E32. Use convergent sequences to prove the theorem that two continuous functions f and g from (X, d) into a metric space are identical if they agree on a dense set in X.

E33. Suppose $f : \mathbb{R} \to \mathbb{R}$. Then we can define $F : \mathbb{R} \to \mathbb{R}^2$ by $F(x) = (x, f(x))$, so that ran(F) is the graph of f.

a) Prove that the following statements are equivalent:

i) f is continuous
ii) F is continuous
iii) The sets $\{(x, y) : y \geq f(x)\}$ and $\{(x, y) : y \leq f(x)\}$ are both closed in $\mathbb{R}^2$.

b) Prove that if f is continuous, then its graph is a closed set in $\mathbb{R}^2$. Give a proof or a counterexample for the converse.

E34. Suppose $X = \{x, y, z, w\}$ is a four point set.

a) Show that there is one and only one metric on X that satisfies the following conditions: $s(x, y) = s(y, z) = s(z, x) = 2$ and $s(x, w) = s(y, w) = s(z, w) = 1$.

b) Show that (X, s) cannot be isometrically embedded into the plane $\mathbb{R}^2$ (with its usual metric).

c) Prove or disprove: (X, s) can be isometrically embedded in (ℓ_2, d), where d is the usual metric on ℓ_2.

E35. Suppose (X, d) is a metric space for which $|X| > 1$ and in which $\emptyset$ and X are the only clopen sets. Prove that $|X| \geq c$.
(*Hint: First prove that there must be a nonconstant continuous function $f : X \to \mathbb{R}$. What can you say about the range of f ?*)

E36. Let X be a <u>finite</u> set and let $C^*(X)$ be the set of all bounded continuous functions from X into $\mathbb{R}$. Let s denote the "uniform metric" on $C^*(X)$ given by $s(f, g) = \sup\{|f(x) - g(x)| : x \in X\}$.

a) Prove that $(C^*(X), s)$ is separable.

b) If $X = \mathbb{N}$, is part a) still true?

E37. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ is continuous that satisfy the functional equation $f(x + y) = f(x) + f(x) - f(x)f(y)$ for all x, y.
(*Hint: let $g(x) = f(x) + 1$. What simpler functional equation does g satisfy? What is $g(x)$ when x is rational?*)

Chapter II Review

Explain why each statement is true, or provide a counterexample.

1. A finite set in a metric space must be closed.

2. For $m, n \in \mathbb{N}$, write $m - n = 2^k z$, where z is an integer not divisible by 2.

$$\text{Let} \begin{cases} d(m,m) = 0 \\ d(m,n) = k \quad \text{if } m \neq n \end{cases}$$

Then d is a pseudometric on $\mathbb{N}$.

3. Consider the set $[1, \infty)$ with the metric $s(x,y) = \frac{3|x-y|}{3 + 3|x-y|}$. Let $\mathbb{N}$ have its usual metric d and define $f : [1, \infty) \to \mathbb{N}$ by $f(x) =$ "the largest integer $\leq x$". Then f is continuous.

4. If N_1 and N_2 are neighborhoods of x in (X, d), then $N_1 \cap N_2$ is also a neighborhood of x.

5. For any open subset O of a metric space (X, d), $\text{int}(\text{cl}(O)) = O$.

6. The metric $d(n,m) = \mid \frac{1}{n} - \frac{1}{m} \mid$ on $\mathbb{N}$ is equivalent to the usual metric on $\mathbb{N}$.

7. Define $f : \mathbb{N} \to \mathbb{R}$ by $f(n) =$ the nth digit of the decimal expansion of π. Then f is continuous.

8. The set of all real numbers with a decimal expansion of the form $x = 0.x_1x_1x_3...x_n010101...$ is dense in $[0, 1]$.

9. There are exactly c countable dense subsets of $\mathbb{R}$.

10. Suppose we measure distances in $\mathbb{R}$ using the metric $d(x,y) = \frac{|x-y|}{1+|x-y|}$. Then the function $\cos : \mathbb{R} \to \mathbb{R}$ is continuous at every point $a \in \mathbb{R}$.

11. A subset A in a pseudometric space (X, d) is dense if and only if $\text{int}\,(X - A) = \emptyset$.

12. Let d_t denote the "taxicab" metric $d_t(x,y) = \sum_{i=1}^{n} |x_i - y_i|$ on $\mathbb{R}^n$. $\mathbb{Q}^n$ is dense in $(\mathbb{R}^n, d_t)$.

13. If B is a countable subset of $\mathbb{R}$, then $\mathbb{R} - B$ is dense in $\mathbb{R}$.

14. If $A \subseteq [0, 1]$ and $\text{cl}\, A \neq [0, 1]$, then $\text{int}\, A \neq \emptyset$.

15. Let $A = \{ \frac{1}{n} : n \in \mathbb{N}\}$. The discrete unit metric produces the same topology on A as the usual metric.

16. In a pseudometric space (X, d), $\text{cl}\, A = \text{cl}\,(X - A)$ if and only if A is clopen.

17. If U is an open set in $\mathbb{R}$ and $U \supseteq \mathbb{Q}$, then $U = \mathbb{R}$.

18. There is a sequence of open sets O_n in $\mathbb{R}$ such that $\bigcap_{n=1}^{\infty} O_n = \mathbb{P}$.

19. If $A \subseteq (X, d)$, then $\operatorname{int} A = X - \operatorname{cl}(X - A)$.

20. There are exactly c continuous functions $f : \mathbb{N} \to \mathbb{N}$.

21. Let $\mathbb{R}$ have the metric $d(x, y) = \frac{|x-y|}{1+|x-y|}$ and let $a_n = \frac{n^2}{n^2+1}$. Then $(a_n) \to 1$ in the space $(\mathbb{R}, d)$.

22. Suppose d_1 is a metric on $\mathbb{R}$ with the following property: for every sequence (r_n) in $\mathbb{R}$

$$(r_n) \to 5 \text{ with respect to } d_1 \text{ if and only if } (r_n) \to 5 \text{ with respect to } d.$$

Then $d_1 \sim d =$ the usual metric on $\mathbb{R}$.

23. Suppose $A \subseteq B \subseteq (X, d)$. If $\operatorname{cl} A = \operatorname{cl} B$, $\operatorname{int} A = \operatorname{int} B$, and $\operatorname{Fr} A = \operatorname{Fr} B$, then $A = B$.

24. Suppose $C([0,1])$ has the metric $d(f, g) = \int_0^1 |f - g|$ and define $\Phi : (C([0,1]), d) \to \mathbb{R}$ by $\Phi(f) = \int_0^1 f$. Then Φ is continuous.

25. Let d be the trivial pseudometric on $\mathbb{R}$. In $(\mathbb{R}, d)$, each real number r is the limit of a sequence of irrational numbers.

26. Suppose $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are continuous and that $f \neq g$. Then there must exist a point $p \in \mathbb{Q}$ where $f(p) \neq g(p)$ and a point $q \in \mathbb{P}$ where $f(q) \neq g(q)$.

27. Suppose $f : (X, d) \to (Y, s)$ is continuous <u>at the point</u> $a \in X$, and suppose O is an open set in Y with $f(a) \in O$. Then $f^{-1}[O]$ is an open set containing a.

28. If every convergent sequence in a metric space (X, d) is eventually constant, then $\mathcal{T}_d = \mathcal{P}(X)$.

29. Let $x \in (X, d)$ Suppose (x_n) is a sequence such that $(f(x_n)) \to f(x)$ for every continuous $f : X \to \mathbb{R}$. Then $(x_n) \to x$.

30. For $f, g \in C([0,1])$, let $d^*(f, g) = \sup\{|f(x) - g(x)| : x \in [0,1]\}$. Let (f_n) be a sequence such that $(f_n) \to f$ in $(C([0,1]), d^*)$, where f is the function $f(x) = x + 2$,
Then there is an $N \in \mathbb{N}$ such that whenever $n \geq N$, we have $f_n(x) \geq x$ for all $x \in [0,1]$.

31. If $f : \mathbb{R} \to \mathbb{R}$ is continuous, then the graph of f is a closed subset of $\mathbb{R}^2$.

32. If the graph of a function $f : \mathbb{R} \to \mathbb{R}$ is closed subset of $\mathbb{R}^2$, then f is continuous.

33. The are exactly c different metrics d on $\mathbb{R}$ for which $\mathcal{T}_d =$ the usual topology on $\mathbb{R}$.

34. In $\mathbb{R}$, the interval $[-2, 1]$ can be written as a countable intersection of open sets.

35. For any $f \in \mathbb{N}^{\mathbb{N}}$, then there is a continuous function $g \in \mathbb{R}^{\mathbb{R}}$ such that $g|\mathbb{N} = f$.

36. Let d be the usual metric on $\mathbb{R}$ and d' the discrete unit metric. Suppose $f:(\mathbb{R},d)\to(\mathbb{N},d')$ is continuous. Then f is constant.

37. If $|X|>1$, then there are infinitely many different metrics d on X for which $\mathcal{T}_d$ is the discrete topology.

38. The space $\mathbb{P}$ of irrational numbers is separable.

39. Suppose (X,d) and (Y,d') are pseudometric spaces and that $f:X\to Y$ is continuous at a. If $d(a,b)=0$, then f is also continuous at b.

40. If A and B are subsets of (X,d), then $\text{int}(A\cap B)=\text{int}(A)\cap\text{int}(B)$.

41. Let d^* be the "max metric" on $\mathbb{R}$ and let d_t be the "taxicab metric" on $\mathbb{R}$. For $x\in\mathbb{R}$, let $f(x)=\cos(x^3)$. Then $f:(\mathbb{R},d^*)\to(\mathbb{R},d_t)$ is continuous.

42. Let d be a pseudometric on the set $X=\{0,1\}$. Then either $\mathcal{T}_d=\{\emptyset,X\}$ or $\mathcal{T}_d=\mathcal{P}(X)$.

43. Suppose $f:\mathbb{R}\to\mathbb{R}$ is continuous and $f(p)=p-\sqrt{2}$ for each <u>irrational</u> p. Then $f(17)\in\mathbb{P}$.

44. A finite set in a metric space must be closed.

45. If $f:\mathbb{R}\to\mathbb{N}$ is continuous and $f(1)=1$, then there must exist an irrational number x for which $f(x)=1$.

46. In a metric space (X,d), it cannot happen that $B_\epsilon(a)=X-B_\epsilon(b)$.

47. The discrete unit metric produces the same topology on $\mathbb{N}$ as the usual metric on $\mathbb{N}$.

48. If D is an uncountable dense subset of $\mathbb{R}$ and C is countable, then $D-C$ is dense in $\mathbb{R}$.

49. Suppose that $f:\ell_2\to\mathbb{R}$ is continuous and that $f(x)=0$ for whenever x is any sequence in ℓ_2 that is eventually 0. Then $f((1,\frac{1}{2},\frac{1}{3},...,\frac{1}{n},...))=0$.

50. If $f:(X,d)\to(Y,d')$ is continuous and onto, and X is separable, then Y is separable.

51. In a pseudometric space (X,d), $\text{cl}\,A=\text{int}\,A$ if and only if A is clopen.

52. There exists a dense subset, D, of $\mathbb{R}$ such that every infinite countable subset of D is dense in $\mathbb{R}$.

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