Introduction

These notes are an introduction to set theory and topology. They are the result of teaching a two-semester course sequence on these topics for many years at Washington University in St. Louis. Typically the students were advanced undergraduate mathematics majors, a few beginning graduate students in mathematics, and some graduate students from other areas that include economics and engineering.

Over time my lecture notes evolved into written outlines for students, then written versions of the more involved proofs. The full set of notes was a project completed during the years 2003-2007 with small revisions thereafter.

The usual background for the material is an introductory undergraduate analysis course, mostly because it provides a solid introduction to Euclidean space $\mathbb{R}^n$ and practice with rigorous arguments – in particular, about continuity. Strictly speaking, however, the material is mostly self-contained. Examples are taken now and then from analysis, but they are not crucial to the logical progression. The only real prerequisite is the level of mathematical interest, maturity and patience needed to handle abstract ideas and to read and write careful proofs. A few very capable students have taken this course before introductory analysis (even, rarely, as university freshmen) and invariably they have commented later on how material eased their way into analysis.

The material on set theory is not done axiomatically. However, we do try to provide some informal insights into why an axiomatization of the subject might be valuable and what some of the most important results are. A student with a good grasp of the set-theoretic material – scattered throughout the notes, but heavily concentrated in Chapters I and VIII – will know all the informal set theory most mathematicians ever need and will be in a good position to continue on to a study of axiomatic set theory.

The topological material is all within the area traditionally labeled "general topology." No topics from algebraic topology are included. This was a conscious choice that reflects my own training and tastes, as well as a conviction that students are usually rushed too quickly through the basics of topology in order to get to "where the action is." It is certainly true that general topology has not been the scene of much research for several decades, and most of the research that does still continue is closely related to set theory and mathematical logic. Nevertheless, general topology contains a set of tools that most mathematicians need, whether for work in analysis or other parts of topology.

Many of those basic tools (such as "compactness" and the "product topology") seem very abstract when a student first meets them. It takes time to develop an ownership of these tools. This includes a sense of their significance, an appropriate "feel" for how they behave, and good technique – in short, all the things necessary to make using a compactness argument, say, into a completely routine tool. I believe this process is often short-circuited in the rush to move students along to algebraic topology. The result then can be an introduction to algebraic topology where many tedious details are (appropriately) omitted and the student is ill-equipped to fill them in – or even to feel confident that the omissions are genuinely routine. In that case, a student can begin to feel that the subject has a vague, hand-waving quality about it.

These notes are designed to give the student the necessary practice and build up intuition. They begin with the more concrete material (metric spaces) and move outward to the more general ideas. The basic notions about topological spaces are introduced in the middle of the study of metric spaces to illustrate the idea of increasing abstraction and to highlight some important properties of metric spaces against a background where these properties fail. The result is an exposition that is not as efficient as it could be if the more abstract definitions were stated in the first place. In particular, many of the basic ideas about metric spaces (Chapter II) are revisited in the introductory chapter on topological spaces (Chapter III).

Just as in any mathematics course, solving problems is essential. There are many exercises in the notes, particularly in the early chapters. They vary in difficulty but it is fair to say that a majority of the problems require some thought. Few, if any, could be genuinely called "trivial." For example, in Chapter I (Sets) there are <u>no</u> problems of the sort:

Prove that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

It is assumed that students are sufficiently sophisticated not to need that sort of drill.

There are "Chapter Reviews" at the end of each chapter. Each review consists of a list of statements, each of which requires either an explanation or a counterexample. Presented with statements whose truth is uncertain, students can develop confidence and intuition, learn to make thoughtful connections and guesses, and build a tool chest of examples and counterexamples. Nearly every review statement requires only an insight, a use of an earlier result in a new situation, or the application of a more abstract result in a concrete situation. For almost every true statement, an appropriate justification consists of at a few sentences at most.

These notes were a "labor of love" over many years and are intended as an aid for students, not as a work for publication. Such originality as there is lies in the selection of material and its organization. Many proofs and exercises have been refashioned but others are more-or-less standard fare drawn from sources some of which are now forgotten. Readers familiar with the material will probably recognize overtones of my predecessors and contemporaries such as Arthur H. Stone, Leonard Gillman, Robert McDowell and Stephen Willard. My thanks to them for all their insights and contributions, and to a few hundred students who have worked with various parts of these notes over the years. Of course, any errors are my own.

The notes are organized into ten chapters (I, II,..., X) and each chapter is divided into sections (1, 2,..., n). Definitions, theorems, and examples are numbered consecutively within each of these sections – for example, Definition 4.1, Theorem 4.2, Theorem 4.3, Example 4.4, ... For example, a reference to Theorem 6.4 refers to the 4$^{\text{th}}$ numbered item in Section 6 of the current chapter. A reference to an item outside the current chapter would include the chapter number: for example, Theorem III.6.4 means the 4$^{\text{th}}$ item in Section 6 of Chapter III.

Exercises are numbered consecutively within each chapter: E1, E2, ... A reference to an exercise outside the current chapter would include the chapter number – for example, Exercise III.E8.

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