# Operator theory and Pick interp on Distinguished Varieties

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# Distinguished Varieties:

Algebraic set $A \subset \mathbb{C}^2$ s.t.

$$A \subset \mathbb{D}^2 \times \mathbb{T}^2 \times \mathbb{E}^2$$

$\mathbb{D} \times \mathbb{D}$ bidisk

$\mathbb{T} \times \mathbb{T}$ torus

$(\mathbb{C} \setminus \overline{\mathbb{D}}) \times (\mathbb{C} \setminus \overline{\mathbb{D}})$ exterior bidisk

**Examples** $V = A \cap \mathbb{D}^2$

1. $w^2 = z^3$ Neil parabola
2. $w^2 = z^2$ Two intersecting disks
3. $w^2 = \dfrac{r^2 - z^2}{1 - r^2 z^2}$ Annuli
4. $\det(\Phi(z) - wI) = 0$

   $\Phi$ matrix rational inner

**Thm** (Agler-McCarthy)

Every distinguished variety arises as

$$\{\det(\Phi(z) - wI) = 0\}$$

where $\Phi$ is rational inner

# Operator Theory (joint with Agler, McCarthy)

## Algebraic isopair —

$(V_1, V_2)$ commuting pure isometries s.t.
$\exists p \in \mathbb{C}[z_1, z_2]$ where

$$p(V_1, V_2) = 0.$$

**Theorem**: An algebraic isopair is annihilated by a squarefree minimal polynomial $p \in \mathbb{C}[z,w]$ and $Z_p = \{(z,w) : p(z,w) = 0\}$ is a distinguished variety

## Several ways to model $V = (V_1, V_2)$

① If $V_1, V_2$ have finite multiplicity, $V \cong (M_t, M_\Phi)$ on $H^2 \otimes \mathbb{C}^N$.

② If $V$ is cyclic, $V \cong (M_{z_1}, M_{z_2})$ on $P^2(\mu)$ space on $A \cap \mathbb{T}^2$

**Def:** An isopair $V$ is **cyclic** if $\exists u \in \mathcal{H}$ s.t.

$$\overline{\mathbb{C}[V]u} = H,$$

and **nearly cyclic** if $\exists$ finite codim invariant subspace $K$ s.t. $V|_K$ is cyclic.

**Def:** Two pairs $V$, $W$ **nearly unitarily equivalent** if $\exists$ finite codim invariant subspaces $K_V$, $K_W$ s.t.

$$V|_{K_V} \cong W, \quad W|_{K_W} \cong V$$

**Thm:** 2 nearly cyclic algebraic isopairs are nearly unitarily equivalent iff they have the same minimal polynomial (and hence associated distinguished variety)

Compare to 1 variable:

all cyclic pure isometries are unitarily equivalent to the forward shift.

$\Rightarrow$ isopairs are nearly one isometry

**Example**: $w^2 = z^3$

$$W = (M_{t^2}, M_{t^3}) \text{ on } H^2$$

is not cyclic. Restriction to $\{f : f'(0) = 0\}$ <u>is</u>.

**Strategy**: Desingularize

$h : S \to Z_p \cap \mathbb{D}^2$

Let $W = (M_{h_1}, M_{h_2})$

on $A^2(S) = \overline{A(S)}^{L^2(\omega)}$.

harmonic measure

**Show** any nearly cyclic $V$ is nearly equivalent to $W$.

# Problem:

**Def:** $\overline{\mathbb{C}[V]\{u_1, \dots, u_k\} = H}$

$\Rightarrow V$ is k-cyclic.

**Q:** Is a nearly k-cyclic $V$ nearly equivalent to k cyclic isopairs?

# Pick interpolation on Distinguished varieties

(joint with Jury, McCullough)

**Setup:** $Z_p \subset \mathbb{D}^2 \cup \mathbb{T}^2 \cup \mathbb{E}^2$

$V = Z_p \cap \mathbb{D}^2$

$p \in \mathbb{C}[z,w]$ minimal defining poly

$\deg p = (n,m)$

## Pick Problem: Given

$(z_1, w_1), \ldots, (z_n, w_n) \in \mathcal{V}$

$\lambda_1, \ldots, \lambda_n \in \mathbb{D}$

$\exists ?\, f : \mathcal{V} \longrightarrow \mathbb{D}$ holomorphic

such that $f(z_j, w_j) = \lambda_j$

Expect: $\exists$ family of pos. kernels $\{K^\alpha\}_{\alpha \in I}$ s.t.

$\exists$ interpolant $f \iff$

$$\forall \alpha \quad \left(1-\lambda_i \bar{\lambda}_j\right) K^\alpha\left((z_i, w_i), (z_j, w_j)\right) \geq 0$$

**Need**: Natural family of kernels

**Thm**: $\exists\, P = (p_1, \dots, p_n) \in \mathbb{C}^{1\times n}[z,w]$,
$\exists\, Q = (q_1, \dots, q_m) \in \mathbb{C}^{1\times m}[z,w]$

s.t. for $(z,w), (\zeta,\eta) \in V$

$$\frac{Q(z,w)Q(\zeta,\eta)^*}{1 - z\bar{\zeta}} = \frac{P(z,w)P(\zeta,\eta)^*}{1 - w\bar{\eta}}$$

## Note:

Thm $\Rightarrow$ representation

$$0 = \det(\Phi(z) - wI)$$

GK proof $\rightsquigarrow$ goes through theory of rational inner functions on $\mathbb{D}^2$

**Def**: $P \in \mathbb{C}^{d \times dn}[z,w]$

$Q \in \mathbb{C}^{d \times dm}[z,w]$

$(P,Q)$ is an admissible pair if

$$\frac{PP^*}{1-w\bar{\eta}} = \frac{QQ^*}{1-z\bar{\zeta}} \text{ on } V$$

Kernels on $\mathcal{V}$

$$K((z,w),(\zeta,\eta)) = \frac{PP^*}{1-z\bar{\zeta}} = \frac{QQ^*}{1-w\bar{\eta}}$$

give rise to Hilbert space $\mathcal{H}$ where $(M_z, M_w)$ is a fin. mult. isopair (and vice versa)

**Thm**: A Pick problem on distinguished variety $\mathcal{V}$ is solvable $\Longleftrightarrow$ $\forall K$ coming from an admissible pair $(P, Q)$

$$(1 - \lambda_i \overline{\lambda_j}) K((z_i, w_i), (z_j, w_j)) \geq 0$$

Proof divided into

① abstract interpolation theorem

② polynomial approximation theorem.

Given a family of kernels $K = \{K^\alpha\}_{\alpha \in I}$

$f \in H^\infty_K(\nu) \iff \exists \rho > 0$

s.t. $(\rho^2 - f(x)f(y)^*)K^\alpha(x,y) \geq 0$

for all $\alpha \in I$

$H^\infty_K(\mathcal{V}) \leftarrow$ Abstract $H^\infty$

Kernels from admissible pairs

**Thm**: If $K =$ Kernels from admissible pairs

then can interpolate $(z_i, w_i) \mapsto \lambda_i$

with $f \in H^\infty_K(\mathcal{V})$, $\|f\| \leq 1$

$\iff \forall k \in K$

$$(1 - \lambda_i \overline{\lambda_j}) k((z_i, w_i), (z_j, w_j)) \geq 0$$

Proof $\rightsquigarrow$ prove even more abstract interpolation theorem for

Agler interpolation family of kernels

② Need $H^\infty_K(V) \cong H^\infty(V)$

iso

**Show**: for $q \in \mathbb{C}[z,w]$

$$\|q\|_{H^\infty_K(V)} = \|q\|_{H^\infty(V)}$$

**Show**: $\|f\|_{H^\infty(V)} \leq \|f\|_{H^\infty_K(V)}$

**Show**: Given $f \in H^\infty(V)$

$\|f\|_{H^\infty(V)} \leq 1$, $\exists\, q_n \in \mathbb{C}[z,w]$

$\|q_n\|_{H^\infty(V)} \leq 1$ s.t.

$q_n \longrightarrow f$ locally uniformly

# Future work

Reduce Pick thm
to checking only
scalar kernels

# References

Algebraic pairs of isometries
J. Operator Theory
(with Agler, McCarthy)

Nevanlinna-Pick interp on dist. var...
J. Funct. Anal.
(with Jury, McCullough)