FUNCTION THEORY ON THE NEIL PARABOLA

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1. INTRODUCTION

Distances on a complex space X which are invariant under biholomorphic maps have played an important role in the geometric approach to complex analysis. One of the oldest such distances is the the Carathéodory pseudodistance c_X ("pseudo" because the distance between two points can be zero). It was introduced by C. Carathéodory in 1926 and is extremely simple to define. The distance between two points x and y is defined to be the largest distance (using the Poincaré hyperbolic distance) that can occur between $f(x)$ and $f(y)$ under a holomorphic map f from X to the unit disk $\mathbb{D} \subset \mathbb{C}$. The Kobayashi pseudodistance k_X, introduced by S. Kobayashi in 1967, is defined in the opposite direction: the "distance" between two points x and y is now the infimum of the (hyperbolic) distance that can occur between two points $a, b \in \mathbb{D}$ for which there is a holomorphic map f from the disk to X mapping a to x and b to y. (Actually, there is a small technicality here—see section 4 for the true definition). A consequence of the Schwarz-Pick lemma on the disk (which says holomorphic self-maps of the disk are distance decreasing in the hyperbolic distance) is the fact that $c_X \leq k_X$.

For the purposes of motivating the present article, let us indulge in a short tangent. An interesting question, because of its geometric implications (including the existence of one dimensional analytic retracts), is for which complex spaces do we have $c_X = k_X$? The most important contribution to this question is by L. Lempert [11]. Lempert's theorem proves the Carathéodory and Kobayashi distances agree on a convex domain. This theorem came as a surprise for a couple of reasons: first, convexity is not a biholomorphic invariant, and second, which is our main point here, *there were not many explicit examples available at the time*[1]. While we cannot remedy the problem of a lack of examples in

[1]The plot thickens on this problem: there is a domain, namely the symmetrized bidisc, in $\mathbb{C}^2$ for which the two distances agree, yet this domain is not biholomorphically equivalent to a convex domain. See [9] for a summary of these results.

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the past, we can attempt to add to the current selection of explicit examples. Many theorems about invariant metrics can be proved in the generality of complex spaces (see [10] for instance) yet curiously there do not seem to be any nontrivial, explicit examples of the Carathéodory distance for a complex space *with a singularity*. Perhaps the simplest complex space with a singularity is the variety contained in the bidisk given by

$$N = \{(z, w) \in \mathbb{D}^2 : z^2 = w^3\}$$

Following [9], we shall call this the *Neil parabola*[2]. In their recent follow-up [9] to their monograph [8], M. Jarnicki and P. Pflug pose the following problem:

Find an effective formula for the Carathéodory distance on the Neil parabola N.

In this paper, we give an answer to this problem (see theorem 2.3). In addition, we compute the infinitesimal Carathéodory pseudodistance for the Neil parabola (see theorem 2.4). As applications, we prove a mixed Carathéodory-Pick interpolation result for which known interpolation theorems do not apply (see theorem 2.7) and we prove a result on extending bounded holomorphic functions on the Neil parabola to the entire bidisk (see theorem 2.9).

The general layout of the rest of the paper is as follows. Motivation and background for the two previously mentioned applications are presented in the following two subsections. In section 2, precise statements of definitions and results are given along with a subsection on preliminary facts about complex analysis on the Neil parabola. The rest of the paper is devoted to proofs. (The locations of specific proofs are given near the corresponding theorem statements in section 2.)

1.1. **A mixed Carathéodory-Pick problem.**

Given n points in the unit disk z_i and n target values w_i also in the unit disk, the well-known theorem of G. Pick [13] says exactly when there exists a holomorphic $F : \mathbb{D} \to \mathbb{D}$ satisfying $F(z_i) = w_i$ (this problem was studied independently by Nevanlinna [12]). In fact, the Schwarz-Pick lemma is just the version of this for two points: z_1, z_2 can be interpolated to w_1, w_2 if and only if

$$\left| \frac{w_1 - w_2}{1 - \bar{w}_1 w_2} \right| \leq \left| \frac{z_1 - z_2}{1 - \bar{z}_1 z_2} \right|$$

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[2]The real curve $y^2 = x^3$ and its variations are referred to as Neil's semi-cubical parabola. Named after William Neil (sometimes spelled Neile), a student of John Wallis, it was the first algebraic curve to have its arc length computed via proto-calculus techniques [15].

Similarly, given n complex numbers $a_0, a_1, \ldots, a_{n-1}$ a well-known theorem of Carathéodory and Fejér [3] says when there exists a holomorphic function $F : \mathbb{D} \to \overline{\mathbb{D}}$ with $a_0, a_1, \ldots, a_{n-1}$ as the first n Taylor coefficients of F.[3] For $n = 2$, this is given again by the (infinitesimal) Schwarz-Pick lemma: a_0 and a_1 can be the first two Taylor coefficients exactly when

$$|a_0|^2 + |a_1| \leq 1$$

The first kind of interpolation problem above is called Nevanlinna-Pick interpolation and the second is called Carathéodory-Fejér interpolation. More modern proofs of these theorems, using ideas from operator theory like the commutant lifting theorem of Sz.-Nagy and Foiaş and reproducing kernel Hilbert spaces (see [4] and [1]), make it possible to study so-called *mixed Carathéodory-Pick problems* wherein the idea is to specify several Taylor coefficients at several points in the disk and determine whether there exists a holomorphic function from the disk to the disk with those properties. However, a restriction imposed in all of the usual mixed Carathéodory-Pick problems is that the Taylor coefficients must be specified sequentially (i.e. one cannot ask to specify the first and third Taylor coefficients at a point without specifying the second as well). For example, these problems do not address an interpolation problem of the following form: given $z_1, z_2, z_3, w_1, w_2 \in \mathbb{D}$, when is there a holomorphic function $F : \mathbb{D} \to \mathbb{D}$ satisfying the following?

$$\begin{aligned}
F(z_1) &= w_1 \\
F(z_2) &= w_2 \\
F'(z_3) &= 0
\end{aligned}$$

(1.1)

In fact, as we shall see, solving the problem (1.1) amounts to computing the Carathéodory distance for the Neil parabola. See theorem 2.7 for the exact statement of our result.

1.2. **Extension of bounded holomorphic functions on the Neil parabola.** The following result is a special case of the work of H. Cartan on Stein Varieties (see [6] page 99). (In fact, we are stating it in almost as little generality as possible.)

Theorem 1.2 (Cartan). *Every holomorphic function on a subvariety V of $\mathbb{D}^2$ is the restriction of a holomorphic function on all of $\mathbb{D}^2$.*

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[3]Using $\overline{\mathbb{D}}$ instead of $\mathbb{D}$ is just a trick used to include the constant unimodular valued functions, because we are really talking about functions in the closed unit ball of $H^\infty(\mathbb{D})$. The same idea applies later on to $\mathcal{O}(\mathbb{D}, \overline{\mathbb{D}})$ (of course this notation has not yet been introduced).

A vast improvement on this theorem (again stated in simple terms) was given by P.L. Polyakov and G.M. Khenkin [14]. They proved using the methods of integral formulas that any subvariety V of $\mathbb{D}^2$ satisfying a certain transversality condition has the property that any bounded holomorphic function on V can be extended to a bounded holomorphic function on all of $\mathbb{D}^2$. In fact, there is a bounded linear operator $T : H^\infty(V) \to H^\infty(\mathbb{D}^2)$ with $Tf \mid_V = f$; in other words, there is some constant C such that for any $f \in H^\infty(V)$

$$(1.3) \qquad\qquad ||Tf||_\infty \leq C||f||_\infty$$

The previously mentioned "transversality condition" applies to the Neil parabola, and therefore any bounded holomorphic function on N can be extended to a bounded holomorphic function on the bidisk.

Related to these ideas is a paper of J. Agler and J.E. McCarthy [2], which gives a description of varieties in the bidisk with the property that bounded holomorphic functions can be extended to the bidisk without increasing their H^∞ norm. The Neil parabola is not such a variety as their results show. This can be seen relatively easily from the fact that the Carathéodory pseudodistance on the Neil parabola is not the restriction of the Carathéodory pseudodistance on the bidisk. Meaning, there is some holomorphic function from N to $\mathbb{D}$ which separates two points of N farther than a function from the bidisk to the disk could. Hence, such a function could not be extended to the bidisk without increasing its norm.

This suggests that extremal functions on the Neil parabola for the Carathéodory pseudodistance might be good candidates for functions which extend "badly" to the bidisk. Indeed, this allows us to give a lower bound of $5/4$ on the constant C in (1.3) for the Neil Parabola. In addition to this we present a simple proof using Agler's Nevanlinna-Pick interpolation theorem for the bidisk that any bounded holomorphic function on the Neil parabola can be extended to a bounded holomorphic function on the bidisk with norm increasing by at most a factor of $\sqrt{2}$ if the function vanishes at the origin and by a factor of $2\sqrt{2}+1$ otherwise. This does not exactly reprove Polyakov and Khenkin's result in our context, since we are not claiming the extension can be given by a linear operator. Nevertheless, it is certainly relevant to their result, is much easier to prove, and provides an explicit bound (see theorem 2.9).

2. Definitions and statements of results

Let us define several important notions for this paper. We shall use $\mathcal{O}(X,Y)$ to denote the set of holomorphic maps from X to Y and $\mathcal{O}(X)$

to denote the set of holomorphic functions from X to $\mathbb{C}$, where X and Y are complex spaces possibly containing singularities (this holds for X below as well).

- Frequent use will be made of the family of holomorphic automorphisms ϕ_α of the unit disk $\mathbb{D} \subset \mathbb{C}$ given by

$$(2.1) \qquad \phi_\alpha(z) = \frac{\alpha - z}{1 - \bar{\alpha}z}$$

where $\alpha \in \mathbb{D}$. Note that ϕ_α is its own inverse function. Sometimes we allow α to be in $\partial\mathbb{D}$, but keep in mind that the resulting ϕ_α is no longer an automorphism of the disk and is instead the constant function $\bar{\alpha}$.

- The *pseudo-hyperbolic distance* on $\mathbb{D}$ is defined to be

$$m(a,b) = \left| \frac{a - b}{1 - \bar{a}b} \right|$$

The *Poincaré distance* on $\mathbb{D}$ is given by $\rho = \tanh^{-1} m$.

- The *Poincaré metric* on the disk, which we shall also denote by ρ, is defined to be

$$\rho(z;v) = \frac{|v|}{1 - |z|^2}$$

for $z \in \mathbb{D}$ and $v \in \mathbb{C}$.

- The *Carathéodory pseudodistance* on X is denoted by c_X and is defined by

$$c_X(x,y) := \sup\{\rho(f(x), f(y)) : f \in \mathcal{O}(X, \mathbb{D})\}$$

If we replace ρ above with m, we get what Jarnicki and Pflug call the *Möbius pseudodistance*:

$$c_X^*(x,y) := \sup\{m(f(x), f(y)) : f \in \mathcal{O}(X, \mathbb{D})\}$$

Due to the simple formula for m and the relation $c_X = \tanh^{-1} c_X^*$, the Möbius pseudodistance is more computationally useful for our purposes, and therefore will be used exclusively in all proofs.

- The *Carathéodory pseudometric* C_X is defined to be

$$C_X(x;v) = \sup\{\rho(f(x); df_x(v)) : f \in \mathcal{O}(X, \mathbb{D})\}$$

for $x \in X$ and $v \in T_x X$, the tangent space of X at x. The Carathéodory pseudometric will often be referred to as the *infinitesimal Carathéodory pseudodistance*.

- Finally, the *Lempert function* for X is denoted $\tilde{k}_X$ and is defined by

$$\tilde{k}_X(x,y) = \inf\{\rho(a,b) : \exists f \in \mathcal{O}(\mathbb{D},X) \text{ with } f(a) = x, f(b) = y\}$$

  where $\tilde{k}_X$ is defined to equal ∞ if the above set over which the infimum is taken is empty. The *Kobayashi pseudodistance* k_X is then defined to be largest pseudodistance bounded by $\tilde{k}_X$.

For more information on and examples of these definitions we refer the reader to [7], [8], [9], and [10].

Recall from the introduction that the Neil parabola is the set

$$N = \{(z,w) \in \mathbb{D}^2 : z^2 = w^3\}$$

The set N is a one-dimensional connected analytic variety in $\mathbb{D}^2$ with a singularity at $(0,0)$. Furthermore, N has a bijective holomorphic parametrization $p : \mathbb{D} \to N$ given by

$$(2.2) \qquad\qquad p(\lambda) := (\lambda^3, \lambda^2)$$

The function $q := p^{-1}$ is continuous on N, holomorphic on $N \setminus \{(0,0)\}$, and can be given by $q(z,w) = z/w$ when $(z,w) \neq (0,0)$ (and $q(0,0) = 0$). For the benefit of those readers unfamiliar with holomorphic functions on a variety with a singularity, we include a discussion of these ideas in the concrete context of the Neil parabola in section 2.1. It is known that the Kobayashi pseudodistance k_N and the Lempert function $\tilde{k}_N$ for N are as simple as possible (see [9]):

$$k_N((a,b),(z,w)) = \tilde{k}_N((a,b),(z,w)) = \rho(q(a,b),q(z,w))$$

On the other hand (and to reiterate our goal in this paper), in [9] the authors lament that despite the simplicity of N an effective formula for the Carathéodory pseudodistance c_N is not known. We propose the following as an effective formula for c_N.

Theorem 2.3 (Carathéodory pseudodistance formula)**.** *Given nonzero* $\lambda, \delta \in \mathbb{D}$, *let*

$$\alpha_0 = \frac{1}{2}\left(\frac{1}{\bar{\lambda}} + \lambda + \frac{1}{\bar{\delta}} + \delta\right)$$

then

$$c_N(p(\lambda), p(\delta)) = \begin{cases} \rho(\lambda^2, \delta^2) & \text{if } |\alpha_0| \geq 1 \\ \rho(\lambda^2 \phi_{\alpha_0}(\lambda), \delta^2 \phi_{\alpha_0}(\delta)) & \text{if } |\alpha_0| < 1 \end{cases}$$

Also, $c_N(p(0), p(\lambda)) = \rho(0, \lambda^2) = \tanh^{-1}|\lambda|^2$.

In particular, it should be noted that if λ and δ have an acute angle between them (i.e. $\operatorname{Re}(\lambda\bar{\delta}) > 0$), then $|\alpha_0| > 1$, and the first formula above gives the distance between $p(\lambda)$ and $p(\delta)$. Also, the theorem shows $k_N \neq c_N$ as one might suspect.

In section 3 we shall reduce the above problem to a maximization problem on the closed unit disk, and in section 4 we solve the maximization problem to yield theorem 2.3. In addition, a slightly nicer form of the above formula will be presented as proposition 4.14.

As will be explained in subsection 2.1, the tangent spaces of N can be identified with subspaces of the tangent spaces of $\mathbb{D}^2$. In particular, for $x = (a, b) \neq (0, 0)$, $T_x N$ is simply the span of the vector $(3a, 2b)$, while the tangent space at the origin of N is two dimensional and therefore equal to all of $\mathbb{C}^2 = T_{(0,0)}\mathbb{D}^2$. We can now present our formula for the Carathéodory pseudometric of N (this is proved in section 5).

Theorem 2.4 (Carathéodory pseudometric formula). *For $v = (v_1, v_2) \in \mathbb{C}^2$, we have*

$$(2.5) \qquad C_N((0,0); v) = \begin{cases} |v_2| & \text{if } |v_2| \geq 2|v_1| \\ \frac{4|v_1|^2 + |v_2|^2}{4|v_1|} & \text{if } |v_2| < 2|v_1| \end{cases}$$

and for $(a, b) \in N$ nonzero and $z \in \mathbb{C}$ we have

$$(2.6) \qquad C_N((a, b); z(3a, 2b)) = \frac{2|b|}{1 - |b|^2}|z|$$

As mentioned in subsection 1.1, a direct consequence of the preceding formulas is the following atypical mixed Carathéodory-Pick interpolation result (see section 6 for the proof).

Theorem 2.7 (Mixed interpolation problem). *Given distinct $z_1, z_2, z_3 \in \mathbb{D}$ and $w_1, w_2 \in \mathbb{D}$, there exists $F \in \mathcal{O}(\mathbb{D}, \mathbb{D})$ with*

$$F(z_i) = w_i \text{ for } i = 1, 2$$
$$F'(z_3) = 0$$

if and only if

$$(2.8) \qquad \rho(w_1, w_2) \leq c_N(p(\phi_{z_3}(z_1)), p(\phi_{z_3}(z_2)))$$

Moreover, if the problem is extremal (i.e. if there is equality in (2.8)), then the solution is unique and is a Blaschke product of order two or three.

The significance of the theorem (which one could say in the way it is stated now practically follows from definitions) is of course that c_N

is directly computable by theorem 2.3. (So, inequality (2.8) is easy to check.)

Finally, in section 7 we prove the following result on extending bounded holomorphic functions from the Neil parabola to the bidisk.

Theorem 2.9 (Bounded analytic extension)**.** *For any $f \in \mathcal{O}(N, \mathbb{D})$ with $f(0,0) = 0$, there exists an extension of f to a function in $\mathcal{O}(\mathbb{D}^2, \sqrt{2}\mathbb{D})$. If $f(0,0) \neq 0$, then f can be extended to $\mathcal{O}(\mathbb{D}^2, (2\sqrt{2}+1)\mathbb{D})$. In addition, there exists a function in $\mathcal{O}(N, \mathbb{D})$ which cannot be extended to a function in $\mathcal{O}(\mathbb{D}^2, r\mathbb{D})$ for $r < 5/4$.*

Here $r\mathbb{D}$ just refers to the disk of radius r.

2.1. **Complex analysis on the Neil parabola.** In this subsection we discuss how to do complex analysis on a variety with a singularity in the concrete setting of the Neil parabola. This is adapted from [9] and [5] (see pages 18-20 and the chapter on tangent spaces) and nothing in this section is by any means new. The most important facts of this subsection are summarized in the two "observations" 2.10 and 2.11.

A function f on N is defined to be holomorphic if at each point $x \in N$, there is a holomorphic function F on a neighborhood U of x in the bidisk which agrees with f on $U \cap N$. Fortunately, we can give a more concrete description of the set of holomorphic functions on N. Given $f \in \mathcal{O}(N)$, the function $h := f \circ p$ (recall p from (2.2)) is an element of $\mathcal{O}(\mathbb{D})$ satisfying $h'(0) = 0$. The reason for this is given an extension, F, of f holomorphic on a neighborhood of $(0,0)$ in $\mathbb{D}^2$, $h = F \circ p$ is holomorphic on a neighborhood of 0 in $\mathbb{D}$. Hence, the derivative $h'(\lambda) = dF_{p(\lambda)}(3\lambda^2, 2\lambda)$ and so $h'(0) = 0$.

Conversely, suppose $h \in \mathcal{O}(\mathbb{D})$ satisfies $h'(0) = 0$. Then, $f := h \circ q$ (recall $q := p^{-1}$) is holomorphic on $N \setminus \{(0,0)\}$ because $F(z, w) = h(z/w)$ is holomorphic on the set $\{(z,w) \in \mathbb{D}^2 : |z| < |w|\}$ which is an open neighborhood of $N \setminus \{(0,0)\}$. To prove f is holomorphic at $(0,0)$, observe first of all that h can be written as an absolutely convergent power series $h(\lambda) = a_0 + a_2\lambda^2 + a_3\lambda^3 + \cdots$ in some (or any) closed disk centered at the origin of radius, say, $r < 1$. Then, for (z, w) with $|z| < 1, |w| < r^3$,

$$F(z, w) := a_0 + a_2 w + a_3 z + a_4 w^2 + a_5 zw + a_6 w^3 + \cdots$$

converges absolutely and extends f (where we are choosing to extend $(z/w)^k$ to a monomial of the form zw^m or w^m—i.e. we want the power of w to be as large as possible).

Let us emphasize the correspondence just proved:

Observation 2.10. *The map given by $f \mapsto f \circ p$ is a bijection from $\mathcal{O}(N)$ to $\{h \in \mathcal{O}(\mathbb{D}) : h'(0) = 0\}$ with inverse given by $h \mapsto h \circ q$, where $p \in \mathcal{O}(\mathbb{D}, N)$ is $p(\lambda) = (\lambda^3, \lambda^2)$ and $q = p^{-1}$.*

Next, we discuss the complex tangent spaces of N. We can define $T_x N$ as a subset of $T_x \mathbb{D}^2 \cong \mathbb{C}^2$ in the following way. Given $v \in \mathbb{C}^2$, $v \in T_x N$ if and only if $dG_x v = 0$ for every holomorphic function G defined in a neighborhood U (in $\mathbb{D}^2$) of x with G identically zero restricted to $U \cap M$. Notice that this definition is designed to make it easy to define the differential of a function $g \in \mathcal{O}(N)$.

If $x = p(\lambda) = (a, b) \neq (0, 0)$ then $T_x N$ is the span of the vector $(3a, 2b)$, because given G, as in the previous paragraph, the function $g := G \circ p$ is identically zero and so $0 = g'(\lambda) = dG_x(3\lambda^2, 2\lambda)$. Hence, $dG_x(3a, 2b) = 0$. On the other hand, $h(z, w) = z^2 - w^3$ vanishes on N and $dh_x v = 0$ if and only if v is a multiple of $(3a, 2b)$.

At the origin $x = (0, 0)$, we have $T_x N = \mathbb{C}^2$, because given G, as above, we have $dG_{(0,0)} = (0, 0)$. This is because the partial derivatives of G at $(0, 0)$ are the coefficients of λ^3 and λ^2 in the identically zero power series for $G(\lambda^3, \lambda^2)$. Let us emphasize the above facts:

Observation 2.11. *The tangent space $T_{(a,b)}N$ at the point $(a, b) \in N$ with $(a, b) \neq (0, 0)$ can be identified with $\{\zeta(3a, 2b) : \zeta \in \mathbb{C}\} \subset \mathbb{C}^2$. The tangent space $T_{(0,0)}N$ can be identified with $\mathbb{C}^2$.*

3. Reduction of theorem 2.3 to a max problem on $\overline{\mathbb{D}}$

As mentioned earlier, we shall compute a formula for c_N^* (which of course gives a formula for c_N).

Because of observation 2.10, we immediately have

$$(3.1) \quad c_N^*(p(\lambda), p(\delta)) = \sup\{m(h(\lambda), h(\delta)) : h \in \mathcal{O}(\mathbb{D}, \mathbb{D}), h'(0) = 0\}$$

As m is invariant under automorphisms of the disk, we may assume $h(0) = 0$ by applying appropriate automorphisms of the disk, since the condition $h'(0) = 0$ is preserved by (post) composition. Then, by the Schwarz lemma, h may be written as $h(\zeta) = \zeta^2 g(\zeta)$ for some $g \in \mathcal{O}(\mathbb{D}, \overline{\mathbb{D}})$. At this stage it is clear that $c_M^*(p(0), p(\lambda)) = |\lambda|^2$. As g varies over all of $\mathcal{O}(\mathbb{D}, \overline{\mathbb{D}})$, the set of pairs $(g(\lambda), g(\delta))$ is just the set of all (A, B) satisfying $m(A, B) \leq m(\lambda, \delta)$. Hence,

$$c_N^*(p(\lambda), p(\delta)) = \sup\{m(\lambda^2 A, \delta^2 B) : A, B \in \mathbb{D} \text{ with } m(A, B) \leq m(\lambda, \delta)\}$$

Since $m(\lambda^2 A, \delta^2 B)$ is the modulus of a holomorphic function in A, the above supremum may be taken over all (A, B) with $m(A, B) = m(\lambda, \delta)$, by the maximum principle. We may safely multiply both A and B a unimodular constant and leave $m(\lambda^2 A, \delta^2 B)$ unchanged. Thus, we can

assume there is some $\alpha \in \mathbb{D}$ such that $A = \phi_\alpha(\lambda)$ and $B = \phi_\alpha(\delta)$ (recall ϕ_α from (2.1)).

By the preceding discussion we have the following formula for c_N^* which gives the desired reduction to a maximization problem.

Proposition 3.2.

$$c_N^*(p(\lambda), p(\delta)) = \max_{\alpha \in \mathbb{D}} m(\lambda^2 \phi_\alpha(\lambda), \delta^2 \phi_\alpha(\delta))$$

In particular, the supremum in (3.1) is attained by some function of the form $h(\zeta) = \zeta^2 \phi_\alpha(\zeta)$ where $\alpha \in \overline{\mathbb{D}}$. Moreover, if h attains the supremum in (3.1) and $h(0) = 0$, then h is of the same form (i.e. $h = \zeta^2 \phi_\alpha$) up to multiplication by a unimodular constant. As we shall see later, either the supremum will be obtained with a unique $\alpha \in \mathbb{D}$ or with any $\alpha \in \partial\mathbb{D}$.

4. Proof of theorem 2.3

To begin, we shall keep λ and δ fixed throughout the section and define a continuous function, smooth except possibly where it is zero, $F : \overline{\mathbb{D}} \to [0, 1)$ by

$$(4.1) \qquad\qquad F(\alpha) := m(\lambda^2 \phi_\alpha(\lambda), \delta^2 \phi_\alpha(\delta))$$

A couple of things to notice about F are

$$F(\alpha) < m(\lambda, \delta) \text{ for all } \alpha \in \overline{\mathbb{D}} \text{ and}$$

$$(4.2) \qquad\qquad F(\alpha) = m(\lambda^2, \delta^2) \text{ for all } \alpha \in \partial\mathbb{D}.$$

As in the statement of theorem 2.3, we let

$$\alpha_0 := \frac{1}{2}\left(\frac{1}{\bar{\lambda}} + \lambda + \frac{1}{\bar{\delta}} + \delta\right)$$

By proposition 3.2, the following two claims (given as lemma 4.6 and lemma 4.11 below) yield theorem 2.3. First, F has no local maximum in $\mathbb{D}$ except possibly α_0. Second, when $|\alpha_0| < 1$, $F(\alpha) \leq F(\alpha_0)$ for all α with $|\alpha| = 1$. Before we prove these facts let us first mention a couple of useful formulas for F whose proofs we defer to the end of the section.

Claim 4.3.

$$(4.4) \quad F(\alpha) = m(\lambda, \delta)\left|\frac{(\lambda + \delta)(\alpha + \lambda\delta\bar{\alpha} - \lambda - \delta) + \lambda\delta(1 - |\alpha|^2)}{(1 + \lambda\bar{\delta})(1 + \lambda\bar{\delta} - \bar{\alpha}\lambda - \alpha\bar{\delta}) - \lambda\bar{\delta}(1 - |\alpha|^2)}\right|$$

$$(4.5) \qquad = m(\lambda, \delta)\left|\frac{1 - (\bar{\alpha} - \bar{\alpha}_0 - \bar{\beta}_2)(\alpha - \alpha_0 + \beta_2)}{1 - (\bar{\alpha} - \bar{\alpha}_0 - \bar{\beta}_1)(\alpha - \alpha_0 + \beta_1)}\right|$$

where

$$\beta_1 := \frac{1}{2}\left(\frac{1}{\bar{\lambda}} - \lambda - \frac{1}{\bar{\delta}} + \delta\right) \quad and$$

$$\beta_2 := \frac{1}{2}\left(\frac{1}{\bar{\lambda}} - \lambda + \frac{1}{\bar{\delta}} - \delta\right)$$

Lemma 4.6. *The function F has no local maximum in $\mathbb{D}$ except possibly at α_0.*

Proof. Using the formula (4.5), it suffices to prove the function given by

$$(4.7) \qquad G(z) = \left(\frac{F(z + \alpha_0)}{m(\lambda, \delta)}\right)^2 = \left|\frac{1 - (\bar{z} - \bar{\beta}_2)(z + \beta_2)}{1 - (\bar{z} - \bar{\beta}_1)(z + \beta_1)}\right|^2$$

has no local max for $|z + \alpha_0| < 1$ except possibly at $z = 0$. Some omitted computations show that G can be written as G_2/G_1 where

$$(4.8) \qquad G_k(z) = 1 + 2|\beta_k|^2 - 2|z|^2 + |z^2 - \beta_k^2|^2$$

for $k = 1, 2$.

Throughout the following, suppose z is a local maximum satisfying $0 < |z + \alpha_0| < 1$. This implies several things:

- $0 < G(z) < 1$,
- z is a critical point for G,
- $\Delta \log G(z) \leq 0$, and
- $\det \text{Hess}(\log G) \geq 0$ at z.

Here Hess denotes the matrix of second partial derivatives. We will prove that all of these conditions cannot be satisfied.

Let us compute all of the derivatives of G_1 and G_2 up to second order. Luckily we can examine G_1 and G_2 simultaneously. Writing $z = x + iy$ we have

$$\partial_z G_k = -2\bar{z} + 2z(\bar{z}^2 - \bar{\beta}_k^2)$$
$$\partial_x G_k = -4x + 4\text{Re}[z(\bar{z}^2 - \bar{\beta}_k^2)]$$
$$\partial_y G_k = -4y - 4\text{Im}[z(\bar{z}^2 - \bar{\beta}_k^2)]$$
$$\partial_{xx}^2 G_k = -4 + 4|z|^2 + 8x^2 - 4\text{Re}\beta_k^2$$
$$\partial_{yy}^2 G_k = -4 + 4|z|^2 + 8y^2 + 4\text{Re}\beta_k^2$$
$$\partial_{xy}^2 G_k = 8xy - 4\text{Im}\beta_k^2$$

Since z is a critical point for G, we have $G_1\partial_z G_2 - G_2\partial_z G_1 = 0$ at z. Neither G_1 nor G_2 vanish at z, and as a result if $\partial_z G_1 = 0$ then

$\partial_z G_2 = 0$. But, $\partial_z G_1$ and $\partial_z G_2$ vanish simultaneously only at 0:

$$\partial_z G_k = -2\bar{z} + 2z(\bar{z}^2 - \bar{\beta_k}^2) = 0$$

for $k = 1, 2$ implies $\bar{z}(\beta_1^2 - \beta_2^2) = 0$, which can only happen if $z = 0$ (because $\beta_1^2 - \beta_2^2 = -(1 - |\lambda|^2)(1 - |\delta|^2)/(\bar{\lambda}\bar{\delta}) \neq 0$). Therefore, at z

$$(4.9) \qquad \frac{G_2}{G_1} = \frac{\partial_z G_2}{\partial_z G_1}, \qquad \frac{\partial_x G_1}{G_1} = \frac{\partial_x G_2}{G_2}, \quad \text{and} \quad \frac{\partial_y G_1}{G_1} = \frac{\partial_y G_2}{G_2}$$

A fact derived from the first of these equations is

$$(4.10) \qquad \left(\frac{\bar{\beta_1}^2}{G_1} - \frac{\bar{\beta_2}^2}{G_2}\right) z^2 = |z|^2(1 - |z|^2)\left(\frac{1}{G_2} - \frac{1}{G_1}\right)$$

and in particular the expression on the left is real.

Using the last two equations in (4.9), we can see that at the critical point z the following equations hold

$$\partial_{xx}^2 \log G = \frac{\partial_{xx}^2 G_2}{G_2} - \frac{\partial_{xx}^2 G_1}{G_1}$$

$$= (-4 + 4|z|^2 + 8x^2)\left(\frac{1}{G_2} - \frac{1}{G_1}\right) + 4\mathrm{Re}\left(\frac{\bar{\beta_1}^2}{G_1} - \frac{\bar{\beta_2}^2}{G_2}\right)$$

$$= -4[(1 - |z|^2)(1 - \mathrm{Re}(z^2/|z|^2)) - 2x^2]\left(\frac{1}{G_2} - \frac{1}{G_1}\right)$$

where the last equality follows from (4.10). Similarly,

$$\partial_{yy}^2 \log G = -4[(1 - |z|^2)(1 + \mathrm{Re}(z^2/|z|^2)) - 2y^2]\left(\frac{1}{G_2} - \frac{1}{G_1}\right)$$

$$\partial_{xy}^2 \log G = 4[2xy + (1 - |z|^2)\mathrm{Im}(z^2/|z|^2)]\left(\frac{1}{G_2} - \frac{1}{G_1}\right)$$

Therefore,

$$\Delta \log G = -8(1 - 3|z|^2)\left(\frac{1}{G_2} - \frac{1}{G_1}\right)$$

and as this must be less than or equal to zero at z, we see that $|z|^2 \leq 1/3$.

Finally, we can show that $\det \mathrm{Hess}(\log G) < 0$, contradicting the fact that z is assumed to be a local maximum. The determinant of the Hessian of the logarithm of G (with the positive factor $16(1/G_2 - 1/G_1)^2$

omitted) is

$$(1 - |z|^2)^2(1 - (\text{Re}(z^2/|z|^2))^2) + 4x^2y^2 - 2|z|^2(1 - |z|^2)$$
$$+ 2(y^2 - x^2)(1 - |z|^2)\text{Re}(z^2/|z|^2)$$
$$- 4x^2y^2 - 4xy(1 - |z|^2)\text{Im}(z^2/|z|^2) - (1 - |z|^2)^2(\text{Im}(z^2/|z|^2))^2$$

Canceling the positive factor $(1-|z|^2)$ and simplifying, we get $-4|z|^2$ which is indeed negative, as promised.

$\square$

Lemma 4.11. *If $|\alpha_0| < 1$, then $F(\alpha) \leq F(\alpha_0)$ for all α with $|\alpha| = 1$.*

Proof. Recall from (4.2) that on the boundary of $\overline{\mathbb{D}}$, F is constant and equal to $m(\lambda^2, \delta^2)$. From equation (4.4) it suffices to prove the inequality

$$\left| \frac{\lambda + \delta}{1 + \bar{\lambda}\delta} \right|^2 \leq \left| \frac{(\lambda + \delta)(\alpha_0 + \lambda\delta\bar{\alpha}_0 - \lambda - \delta) + \lambda\delta(1 - |\alpha_0|^2)}{(1 + \lambda\bar{\delta})(1 + \lambda\bar{\delta} - \bar{\alpha}_0\lambda - \alpha_0\bar{\delta}) - \lambda\bar{\delta}(1 - |\alpha_0|^2)} \right|^2$$

Assuming the left hand side above is nonzero (which we can), it suffices to prove

$$(4.12) \qquad \begin{aligned} &\left| (\alpha_0 + \lambda\delta\bar{\alpha}_0 - \lambda - \delta) + \lambda\delta\frac{(1 - |\alpha_0|^2)}{\lambda + \delta} \right|^2 \\ &- \left| (1 + \lambda\bar{\delta} - \bar{\alpha}_0\lambda - \alpha_0\bar{\delta}) - \lambda\bar{\delta}\frac{(1 - |\alpha_0|^2)}{1 + \lambda\bar{\delta}} \right|^2 \geq 0 \end{aligned}$$

If we think of the left hand side as $|A + B|^2 - |C + D|^2 = |A|^2 - |C|^2 + 2\text{Re}(A\bar{B} - C\bar{D}) + |B|^2 - |D|^2$, then first of all $|A|^2 - |C|^2$ equals

$$|\alpha_0 + \lambda\delta\bar{\alpha}_0 - \lambda - \delta|^2 - |1 + \lambda\bar{\delta} - \bar{\alpha}_0\lambda - \alpha_0\bar{\delta}|^2 = -(1 - |\alpha_0|^2)(1 - |\lambda|^2)(1 - |\delta|^2)$$

and using the identities

$$\alpha_0 + \lambda\delta\bar{\alpha}_0 - (\lambda + \delta) = \frac{\bar{\lambda} + \bar{\delta}}{2\bar{\lambda}\bar{\delta}}(1 + |\lambda\delta|^2)$$

$$1 + \lambda\bar{\delta} - \bar{\alpha}_0\lambda - \alpha_0\bar{\delta} = -\frac{1 + \bar{\lambda}\delta}{2\bar{\lambda}\delta}(|\lambda|^2 + |\delta|^2)$$

we get $2\text{Re}(A\bar{B} - C\bar{D}) = (1 - |\alpha_0^2|)(1 - |\lambda|^2)(1 - |\delta|^2)$.

Also, using the identity

$$(4.13) \qquad |1 + a\bar{b}|^2 - |a + b|^2 = (1 - |a|^2)(1 - |b|^2)$$

we see that $|B|^2 - |D|^2$ equals

$$|\lambda\delta|^2(1 - |\alpha_0|^2)^2\frac{(1 - |\lambda|^2)(1 - |\delta|^2)}{|\lambda + \delta|^2|1 + \lambda\bar{\delta}|^2}$$

Summing this all up, we see that proving (4.12) amounts to showing

$$|\lambda\delta|^2(1-|\alpha_0|^2)^2\frac{(1-|\lambda|^2)(1-|\delta|^2)}{|\lambda+\delta|^2|1+\lambda\bar\delta|^2} \geq 0$$

which is certainly true.

$\square$

This concludes the proof of theorem 2.3. As promised, a slightly nicer formula for $c_N^*(p(\lambda), p(\delta))$ is

Proposition 4.14. *If $\lambda, \delta \in \mathbb{D}$ are nonzero, then*

$$c_N^*(p(\lambda), p(\delta)) = \begin{cases} m(\lambda^2, \delta^2) & \text{if } |\alpha_0| \geq 1 \\ m(\lambda, \delta)\frac{1+|\beta_2|^2}{1+|\beta_1|^2} & \text{if } |\alpha_0| < 1 \end{cases}$$

This follows from proposition 3.2, the definition of F (namely equation (4.1)), formula (4.5) for F, and the above lemmas.

We conclude this section with the proof of claim 4.3.

Proof of Claim 4.3: We start from equation (4.1). Observe that

$$
\begin{aligned}
F(\alpha) &= \left| \frac{\lambda^2\frac{\alpha-\lambda}{1-\bar\alpha\lambda} - \delta^2\frac{\alpha-\delta}{1-\bar\alpha\delta}}{1 - \lambda^2\bar\delta^2\frac{\alpha-\lambda}{1-\bar\alpha\lambda}\frac{\bar\alpha-\bar\delta}{1-\alpha\bar\delta}} \right| \\
&= \left| \frac{\lambda^2(\alpha-\lambda)(1-\bar\alpha\delta) - \delta^2(\alpha-\delta)(1-\bar\alpha\lambda)}{(1-\bar\alpha\lambda)(1-\alpha\bar\delta) - \lambda^2\bar\delta^2(\alpha-\lambda)(\bar\alpha-\bar\delta)} \right| \\
&= \left| \frac{\alpha(\lambda^2-\delta^2) - (\lambda^3-\delta^3) - |\alpha|^2\lambda\delta(\lambda-\delta) + \lambda\delta(\lambda^2-\delta^2)\bar\alpha}{1-\lambda^3\bar\delta^3 - \bar\alpha\lambda(1-\lambda^2\bar\delta^2) - \alpha\bar\delta(1-\lambda^2\bar\delta^2) + |\alpha|^2\lambda\bar\delta(1-\lambda\bar\delta)} \right|
\end{aligned}
$$

(4.15)
$$
\begin{aligned}
&= m(\lambda, \delta)\left| \frac{\alpha(\lambda+\delta) - (\lambda^2+\lambda\delta+\delta^2) - |\alpha|^2\lambda\delta + \lambda\delta(\lambda+\delta)\bar\alpha}{1+\lambda\bar\delta+\lambda^2\bar\delta^2 - \bar\alpha\lambda(1+\lambda\bar\delta) - \alpha\bar\delta(1+\lambda\bar\delta) + |\alpha|^2\lambda\bar\delta} \right| \\
&= m(\lambda, \delta)\left| \frac{\alpha(\lambda+\delta) + \lambda\delta(\lambda+\delta)\bar\alpha - (\lambda+\delta)^2 + \lambda\delta(1-|\alpha|^2)}{(1+\lambda\bar\delta)^2 - \bar\alpha\lambda(1+\lambda\bar\delta) - \alpha\bar\delta(1+\lambda\bar\delta) - (1-|\alpha|^2)\lambda\bar\delta} \right|
\end{aligned}
$$

and from here it is easy to get (4.4).

Secondly, to prove (4.5), we start from (4.15):

$$
\begin{aligned}
F(\alpha) &= m(\lambda, \delta)\left| \frac{\lambda\delta - (\lambda\delta\bar\alpha - (\lambda+\delta))(\alpha - (\lambda+\delta))}{\lambda\bar\delta - (\bar\alpha\lambda - (1+\lambda\bar\delta))(\alpha\bar\delta - (1+\lambda\bar\delta))} \right| \\
&= m(\lambda, \delta)\left| \frac{1 - (\bar\alpha - (1/\delta+1/\lambda))(\alpha - (\lambda+\delta))}{1 - (\bar\alpha - (1/\lambda+\bar\delta))(\alpha - (1/\bar\delta+\lambda))} \right|
\end{aligned}
$$

and this equals (4.5) because of the identities:

$$\begin{aligned}
\bar{\alpha}_0 + \bar{\beta}_2 &= \frac{1}{\lambda} + \frac{1}{\delta} \\
\alpha_0 - \beta_2 &= \lambda + \delta \\
\bar{\alpha}_0 + \bar{\beta}_1 &= \frac{1}{\lambda} + \bar{\delta} \\
\alpha_0 - \beta_1 &= \lambda + \frac{1}{\bar{\delta}}
\end{aligned}$$

$\square$

5. THE INFINITESIMAL CARATHÉODORY PSEUDODISTANCE

In this section we prove theorem 2.4, our formula for the Carathéodory pseudometric.

The Carathéodory pseudometric at the origin and a vector $v = (v_1, v_2) \in \mathbb{C}^2$ is

$$C_N((0,0); v) = \sup\{|df_{(0,0)}v| : f \in \mathcal{O}(N, \mathbb{D}) \text{ and } f(0,0) = 0\}$$

Any f as above satisfies $f(\lambda^3, \lambda^2) = \lambda^2 g(\lambda)$ for some $g \in \mathcal{O}(\mathbb{D}, \overline{\mathbb{D}})$ (see the beginning of section 3). Also, the partial derivative of f with respect to the first variable at the origin is just $g'(0)$ and the partial derivative of f with respect to the second variable at the origin is $g(0)$ (see section 2.1). Therefore,

$$C_N((0,0); v) = \sup\{|v_1 g'(0) + v_2 g(0)| : g \in \mathcal{O}(\mathbb{D}, \overline{\mathbb{D}})\}$$

The set of pairs $(g'(0), g(0))$ as g varies over $\mathcal{O}(\mathbb{D}, \overline{\mathbb{D}})$ is really just the pairs (A, B) where $|A| + |B|^2 \leq 1$, by the Schwarz-Pick Lemma. With suitable choices for the arguments of A and B, we can reduce the problem to maximizing $|v_1|s + |v_2|t$ over all $s, t \in [0, 1]$ satisfying $s + t^2 \leq 1$. The function we are maximizing is linear, so the maximum occurs on the boundary. Therefore, the problem is just a matter of finding the maximum of $|v_1|(1 - t^2) + |v_2|t$ for $0 \leq t \leq 1$. So, by calculus,

$$C_N((0,0); v) = \begin{cases} |v_2| & \text{if } |v_2| \geq 2|v_1| \\ \frac{4|v_1|^2 + |v_2|^2}{4|v_1|} & \text{if } |v_2| < 2|v_1| \end{cases}$$

as desired.

Next, let $x = (a, b) \in N \setminus \{(0,0)\}$ and define $v = (3a, 2b)$. The Carathéodory pseudometric at (a, b) is

$$C_N(x; v) = \sup \left\{ \frac{|df_x v|}{1 - |f(x)|^2} : f \in \mathcal{O}(N, \mathbb{D}) \right\}$$

If we set $\lambda = a/b$ and $h = f \circ p$, then $v = \lambda(3\lambda^2, 2\lambda)$ and since $df_x(3\lambda^2, 2\lambda) = h'(\lambda)$ we see that

$$C_N(x; v) = |\lambda| \sup\{\rho(h(\lambda); h'(\lambda)) : h \in \mathcal{O}(\mathbb{D}, \mathbb{D}) \text{ and } h'(0) = 0\}$$

By (post) composing h with an automorphism of the unit disk (which is allowed by invariance properties of ρ), we can assume $h(0) = 0$ and therefore h has the form $h(\zeta) = \zeta^2 g(\zeta)$ for some $g \in \mathcal{O}(\mathbb{D}, \overline{\mathbb{D}})$. Hence,

$$C_N(x; v) = |\lambda| \sup \left\{ \frac{|\lambda^2 g'(\lambda) + 2\lambda g(\lambda)|}{1 - |\lambda|^4 |g(\lambda)|^2} : g \in \mathcal{O}(\mathbb{D}, \overline{\mathbb{D}}) \right\}$$

Like before, $(g'(\lambda), g(\lambda))$ varies over all pairs (A, B) satisfying $|A|(1 - |\lambda|^2) \leq 1 - |B|^2$. This reduces the problem to maximizing

$$\frac{|\lambda|^2 s + 2|\lambda| t}{1 - |\lambda|^4 t^2}$$

over the set of non-negative s, t satisfying $t^2 + s(1 - |\lambda|^2) \leq 1$. It is easy to check that the maximum always occurs when $t = 1$ and $s = 0$. Since $\lambda^2 = b$ we see that

$$C_N(x; v) = \frac{2|b|}{1 - |b|^2}.$$

6. Proof of theorem 2.7

By precomposing all functions with ϕ_{z_3} we may assume $z_3 = 0$ in theorem 2.7. Then, all functions of interest will correspond to functions in $\mathcal{O}(N, \mathbb{D})$, and therefore it is clear that if there is a function $h \in \mathcal{O}(\mathbb{D}, \mathbb{D})$ which satisfies the $h'(0) = 0, h(z_i) = w_i$ for $i = 1, 2$, then the inequality (2.8) holds (by theorem 2.3 and the definition of Carathéodory pseudodistance).

On the other hand, if the inequality (2.8) holds (again with $z_3 = 0$), then pick a function $f \in \mathcal{O}(N, \mathbb{D})$ with

$$\rho(f(p(z_1)), f(p(z_2))) = c_N(p(z_1), p(z_2))$$

(we know such a function exists by the formula for c_N) and then set $h := f \circ p \in \mathcal{O}(\mathbb{D}, \mathbb{D})$. The function h satisfies $\rho(w_1, w_2) \leq \rho(h(z_1), h(z_2))$ and by composing h with an appropriate function we can find a function $F \in \mathcal{O}(\mathbb{D}, \mathbb{D})$ with $F(z_1) = w_1$, $F(z_2) = w_2$, and $F'(0) = 0$.

To prove the last part of theorem 2.7, suppose F satisfies the interpolation problem and equality in (2.8). Then, $h := \phi_{F(0)} \circ F$ satisfies equality as well. Hence, when

$$\alpha_0 := \frac{1}{2} \left(\frac{1}{\bar{z}_1} + z_1 + \frac{1}{\bar{z}_2} + z_2 \right)$$

is in the disk, $h(\lambda)$ is of the form $\mu\lambda^2\phi_{\alpha_0}(\lambda)$ where μ is a unimodular constant, and when $\alpha_0 \notin \mathbb{D}$, $h(\lambda)$ is of the form $\mu\lambda^2$ (again with $\mu \in \partial\mathbb{D}$). But, μ and $F(0)$ are uniquely determined by the fact that $w_i = \phi_{F(0)}(h(z_i))$ for $i = 1, 2$ since $h(z_1)$ and $h(z_2)$ must be distinct. So, there exists a unique automorphism of the disk ψ such that

$$F(\lambda) = \begin{cases} \psi(\lambda^2\phi_{\alpha_0}(\lambda)) & \text{if } \alpha_0 \in \mathbb{D} \\ \psi(\lambda^2) & \text{if } \alpha_0 \notin \mathbb{D} \end{cases}$$

In the first case, F is a Blaschke product of order three and in the second a Blaschke product of order two.

7. Proof of Extension Theorem

In this section we prove Theorem 2.9.

First, we need to define a few basic notions. Let X be a set. A self-adjoint function $F : X \times X \to \mathbb{C}$ (i.e. $F(x, y) = \overline{F(y, x)}$) is *positive semi-definite* if for every positive integer n and every finite subset $\{x_1, x_2, \ldots, x_n\} \subset X$ the $n \times n$ matrix with entries $F(x_i, x_j)$ is positive semi-definite. For example, by the Pick interpolation theorem the function $F : \mathbb{D} \times \mathbb{D} \to \mathbb{C}$ given by

$$F(\lambda, \delta) = \frac{1 - g(\lambda)\overline{g(\delta)}}{1 - \lambda\bar{\delta}}$$

is positive semi-definite for any $g \in \mathcal{O}(\mathbb{D}, \overline{\mathbb{D}})$.

The Pick interpolation theorem on the bidisk (see [1] page 180) can be stated as a theorem about extensions of bounded analytic functions in the following way. Given a subset X of the bidisk, and a function $\psi : X \to \mathbb{D}$ there exists $\Psi \in \mathcal{O}(\mathbb{D}^2, \mathbb{D})$ with $\Psi \mid_X = \psi$ if and only if there exist positive semi-definite functions Δ and Γ on $X \times X$ such that for each $z = (z_1, z_2), w = (w_1, w_2) \in X$

$$1 - \psi(z)\overline{\psi(w)} = \Gamma(z, w)(1 - z_1\bar{w}_1) + \Delta(z, w)(1 - z_2\bar{w}_2)$$

We should mention that the portion of this theorem which we shall use (namely sufficiency) has a quite simple proof—it is an application of the so-called "lurking isometry" technique.

To prove theorem 2.9 suppose $f \in \mathcal{O}(N, \mathbb{D})$ and $f(0, 0) = 0$. Then, as in earlier arguments, $(f \circ p)(\lambda) = f(\lambda^3, \lambda^2) = \lambda^2 g(\lambda)$ for some $g \in \mathcal{O}(\mathbb{D}, \overline{\mathbb{D}})$. For any $\delta, \lambda \in \mathbb{D}$, we have

$$2 - f(p(\lambda))\overline{f(p(\delta))} = (1 - \lambda^3\bar{\delta}^3)$$
$$+ \left(1 + \lambda^2\bar{\delta}^2\frac{1 - g(\lambda)\overline{g(\delta)}}{1 - \lambda\bar{\delta}} + \frac{\lambda^3\bar{\delta}^3 g(\lambda)\overline{g(\delta)}}{1 - \lambda^2\bar{\delta}^2} \right)(1 - \lambda^2\bar{\delta}^2)$$

Therefore, for $z = (z_1, z_2), w = (w_1, w_2) \in N$

(7.1) $\qquad 2 - f(z)\overline{f(w)} = \Gamma(z, w)(1 - z_1\bar{w}_1) + \Delta(z, w)(1 - z_2\bar{w}_2)$

where $\Gamma(z, w) = 1$ and

$$\Delta(z, w) = 1 + z_1\bar{w}_1 \frac{1 - g(q(z))\overline{g(q(w))}}{1 - q(z)\overline{q(w)}} + \frac{z_2\bar{w}_2 g(q(z))\overline{g(q(w))}}{1 - z_1\bar{w}_1}$$

(recall $q(z) = z_1/z_2$ for $z \neq (0, 0)$ and $q(0, 0) = 0$). Now, Γ is clearly positive semi-definite, and Δ is positive semi-definite because of the fact that positive semi-definite functions are closed under addition and multiplication (by the Schur product theorem) and by the Pick interpolation theorem on the disk (applied to g). This proves f has an extension to the bidisk with supremum norm at most $\sqrt{2}$ (by dividing through (7.1) by 2).

To prove any holomorphic function $f \in \mathcal{O}(N, \mathbb{D})$ (regardless of its value at the origin) can be extended to the bidisk with supremum norm at most $2\sqrt{2} + 1$, simply apply the result just proved to $(f - f(0))/2$.

Finally, the function

$$h(\lambda) = \lambda^2 \frac{0.5 - \lambda}{1 - 0.5\lambda}$$

corresponds to a function $f \in \mathcal{O}(N, \mathbb{D})$ with $f(\lambda^3, \lambda^2) = h(\lambda)$. The partial derivatives of f at $(0, 0)$ are just the coefficients of λ^3 and λ^2 in the power series for h; i.e. they are -0.75 and 0.5. Suppose F is a bounded holomorphic extension of f to the bidisk with sup norm R. Then, by the Schwarz lemma on the bidisk

$$0.75/R + 0.5/R \leq 1$$

which implies $R \geq 5/4$, as desired.

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