A refined Agler decomposition and geometric applications

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Pick's theorem

Theorem

Suppose $f : \mathbb{D} \to \mathbb{D}$ holomorphic.
Then, for any n points $z_1, \ldots, z_n$, the $n \times n$ matrix

$$M = \left(\frac{1 - f(z_j)\overline{f(z_k)}}{1 - z_j \bar{z}_k} \right)$$

is positive semi-definite.

Conversely, a function $f : X \to \mathbb{D}$ on a finite set $X = \{z_1, \ldots, z_n\}$, with M positive semi-definite extends to a holomorphic function $f : \mathbb{D} \to \mathbb{D}$.

Positive semi-definite functions

Definition

A function $K : S \times S \to \mathbb{C}$ is positive semi-definite if for every finite subset X of S,

$$(K(z,w))_{z,w \in X}$$

is positive semi-definite.

The Fundamental Theorem of Positive Semi-definite functions

If $K : S \times S \to \mathbb{C}$ is positive semi-definite, there exists a Hilbert space $\mathcal{H}$ and elements $K_z \in \mathcal{H}$ for each $z \in S$ such that

$$K(z,w) = \langle K_w, K_z \rangle$$

Pick's theorem

$$f : \mathbb{D} \to \mathbb{D} \text{ holomorphic} \implies \frac{1 - f(z)\overline{f(w)}}{1 - z\bar{w}} \text{ is positive semi-definite}$$

- Provides an opening for Hilbert space methods to prove function theory results.
- Example: Sarason's approach to the Julia-Carathéodory theorem.

Agler decomposition

$\mathbb{D}^2 = \mathbb{D} \times \mathbb{D}$

Agler's theorem

If $f : \mathbb{D}^2 \to \mathbb{D}$ is holomorphic, then there exist positive semi-definite functions K_1, K_2 on $\mathbb{D}^2$ such that

$$1 - f(z)\overline{f(w)} = (1 - z_1\bar{w}_1)K_1(z, w) + (1 - z_2\bar{w}_2)K_2(z, w)$$

(Conversely, if such a relation holds on a subset of $\mathbb{D}^2$, the relation extends to all of $\mathbb{D}^2$.)

- Can this be used to prove results in function theory on the bidisk via Hilbert space methods?
- Yes!: Ball-Bolotnikov on boundary interpolation, Agler-McCarthy-Young used these ideas to study Julia-Carathéodory problems, and generalizations of Loewner's theorem on operator monotone functions.

Refined Pick theorem

Hermitian-symmetric Pick Theorem

If $f : \mathbb{D} \to \mathbb{D}$ is holomorphic, then for any $z_1, \ldots, z_n, w_1, \ldots, w_n \in \mathbb{D}$ and any $v_1, \ldots, v_n \in \mathbb{C}$

$$\left| \sum_{j,k} v_j v_k \frac{f(z_j) - f(w_k)}{z_j - w_k} \right|^2$$

$$\leq \sum_{j,k} v_j \bar{v}_k \frac{1 - f(z_j)\overline{f(z_k)}}{1 - z_j \bar{z}_k} \sum_{j,k} v_j \bar{v}_k \frac{1 - f(w_j)\overline{f(w_k)}}{1 - w_j \bar{w}_k}$$

- RHS is "Hermitian." LHS is "symmetric."
- Due to de Branges-Rovnyak?
- Inequalities of this type are common in univalent function theory.

Hermitian-symmetric Pick theorem

So,

$$K(z,w) = \frac{1 - f(z)\overline{f(w)}}{1 - z\bar{w}}$$

is a positive semi-definite function, and

$$L(z,w) = \frac{f(z) - f(w)}{z - w}$$

is a "symmetric holomorphic kernel function."
Our inequality takes the form

$$\Big|\sum_{j,k} v_j v_k L(z_j, w_k)\Big|^2 \leq \sum_{j,k} v_j \bar{v}_k K(z_j, z_k) \sum_{j,k} v_j \bar{v}_k K(w_j, w_k)$$

Hermitian-symmetric Agler decomposition

Refined Agler decomposition (GK)

If $f : \mathbb{D}^2 \to \mathbb{D}$ is holomorphic,

- $\exists K_1, K_2$ on $\mathbb{D}^2$, positive semi-definite and
- $\exists$ holomorphic kernels $L_1, L_2 : \mathbb{D}^2 \times \mathbb{D}^2 \to \mathbb{C}$, such that
- $1 - f(z)\overline{f(w)} = (1 - z_1 \bar{w}_1) K_1(z, w) + (1 - z_2 \bar{w}_2) K_2(z, w)$
- $f(z) - f(w) = (z_1 - w_1) L_1(z, w) + (z_2 - w_2) L_2(z, w)$
- there is a Hermitian-symmetric inequality between K_1 and L_1, and between K_2 and L_2.

In particular, $|L_1(z, w)|^2 \leq K_1(z, z) K_1(w, w)$, which implies

$$\left| \frac{\partial f}{\partial z_1}(z) \right| = |L_1(z, z)| \leq K_1(z, z)$$

Why am I interested in this decomposition?

- Shows further surprising similarities between function theory on $\mathbb{D}$ and $\mathbb{D}^2$.
- (My) Proof uses a detailed Agler decomposition for rational inner functions on $\mathbb{D}^2$. Comes from work of Cole-Wermer or Geronimo-Woerdeman.
- It's not clear that there is an easier proof.
- Application: a new proof of a theorem of Guo et al, related to holomorphic retracts on the polydisk.

Rational inner functions

In one variable, rational inner function = finite Blaschke product.

$$\prod_{j=1}^{n} \frac{z - a_j}{1 - \bar{a}_j z} = \frac{z^n \overline{p(1/\bar{z})}}{p(z)} = \frac{\tilde{p}(z)}{p(z)}$$

In two variables,

$$\text{regular rational inner function } = \frac{\tilde{p}(z_1, z_2)}{p(z_1, z_2)}$$

where $p \in \mathbb{C}[z_1, z_2]$ has no zeros in $\overline{\mathbb{D}^2}$, and $\tilde{p}(z_1, z_2) = z_1^n z_2^m \overline{p(1/\bar{z}_1, 1/\bar{z}_2)}$.

Decompositions for rational inner functions

One variable:

Christoffel-Darboux formula

$$\frac{p(z)\overline{p(w)} - \tilde{p}(z)\overline{\tilde{p}(w)}}{1 - z\bar{w}} = \sum_{j=1}^{n} A_j(z)\overline{A_j(w)}$$

where $A_1, \ldots, A_n \in \mathbb{C}[z]$.

Two variables: $z = (z_1, z_2), w = (w_1, w_2)$

Cole-Wermer formula

$$\begin{aligned} p(z)\overline{p(w)} - \tilde{p}(z)\overline{\tilde{p}(w)} &= (1 - z_1\bar{w}_1)\sum_{j=1}^{n} A_j(z)\overline{A_j(w)} \\ &\quad + (1 - z_2\bar{w}_2)\sum_{j=1}^{m} B_j(z)\overline{B_j(w)} \end{aligned}$$

where $A_j, B_j \in \mathbb{C}[z_1, z_2]$.

Reflection of formulas

Both of the previous formulas can be reflected!

One variable

$$\frac{\tilde{p}(z)p(w) - p(z)\tilde{p}(w)}{z - w} = \sum_{j=1}^{n} \tilde{A}_j(z)A_j(w)$$

Two variables

$$\begin{aligned}\tilde{p}(z)p(w) - p(z)\tilde{p}(w) &= (z_1 - w_1)\sum_{j=1}^{n} \tilde{A}_j(z)A_j(w) \\ &\quad + (z_2 - w_2)\sum_{j=1}^{m} \tilde{B}_j(z)B_j(w)\end{aligned}$$

Approximation

- Holomorphic functions $f : \mathbb{D} \to \mathbb{D}$ or $f : \mathbb{D}^2 \to \mathbb{D}$ can be approximated locally uniformly by rational inner functions.
- Refined Pick and Agler theorems follow by approximation.

Application

The following theorem is due to Kunyu Guo, Hansong Huang, and Kai Wang.

Theorem

Let $V \subset \mathbb{D}^{n+1}$ and suppose $z_{n+1}|_V$ has a non-trivial, norm 1, holomorphic extension to $\mathbb{D}^{n+1}$. Then, there exists $f : \mathbb{D}^n \to \mathbb{D}$ holomorphic such that

$$V \subset \{(z, f(z)) : z \in \mathbb{D}^n\}$$

- Original proof involves interesting use of one variable Denjoy-Wolff theorem.
- Used to build on work of Agler-McCarthy on subvarieties of the bidisk with the "norm preserving holomorphic extension property."

The Guo-Huang-Wang Theorem

- Nice application: the fixed point set of a holomorphic mapping $G : \mathbb{D}^n \to \mathbb{D}^n$, is the graph of a holomorphic function (of less than n variables).
- New proof of Heath and Suffridge's characterization of holomorphic retracts of the polydisk.
- "$V \subset \mathbb{D}^n$ is a retract" means there is a holomorphic $\rho : \mathbb{D}^n \to \mathbb{D}^n$ such that $\rho \circ \rho = \rho$, $\rho(\mathbb{D}^n) = V$.
- Key aspect of using the refined Agler decomposition in my proof: use positive kernels to control derivatives! (and use the Schwarz lemma)

Recap

- Operator related function theory on the bidisk bears a close similarity to the theory on the disk.
- The "refined Agler decomposition" is an application of the extremely detailed Agler decomposition for rational inner functions.
- The decomposition has applications to geometric function theory on the polydisk.

FIN

Subliminal proof

Given $F : \mathbb{D}^2 \to \mathbb{D}$ holomorphic. Consider the set

$$V = \{(z_1, z_2) : F(z_1, z_2) = z_2\} \text{ Is this a graph?}$$

Assume $\varnothing \neq V \neq \mathbb{D}^2$.

- If $\frac{\partial F}{\partial z_2} = 1$ at point of V, Schwarz lemma $\implies$ $F(z_1, z_2)$ depends only on z_2.
- Assume $\frac{\partial F}{\partial z_2} \neq 1$ at every point of V.

Subliminal proof continued

Given $F : \mathbb{D}^2 \to \mathbb{D}$ holomorphic. Consider the set

$$V = \{(z_1, z_2) : F(z_1, z_2) = z_2\} \text{ Is this a graph?}$$

Assume $\varnothing \neq V \neq \mathbb{D}^2$. $\frac{\partial F}{\partial z_2} \neq 1$ on V.

- Refined Agler decomposition:

$$1 - F(z_1, z_2)\overline{F(w_1, w_2)} = (1 - z_1 \bar{w}_1) K_1(z, w) + (1 - z_2 \bar{w}_2) K_2(z, w)$$

$$F(z_1, z_2) - F(w_1, w_2) = (z_1 - w_1) L_1(z, w) + (z_2 - w_2) L_2(z, w)$$

$$|L_1(z, w)|^2 \leq K_1(z, z) K_1(w, w)$$

- Restrict to V:

$$1 - z_2 \bar{w}_2 = (1 - z_1 \bar{w}_1) K_1 + (1 - z_2 \bar{w}_2) K_2$$

$$z_2 - w_2 = (z_1 - w_1) L_1 + (z_2 - w_2) L_2$$

Subliminal proof continued

- Restrict to V:

$$1 - z_2\bar{w}_2 = (1 - z_1\bar{w}_1)K_1 + (1 - z_2\bar{w}_2)K_2$$

$$z_2 - w_2 = (z_1 - w_1)L_1 + (z_2 - w_2)L_2$$

- If $K_2 = 1$ then $K_1 = 0$, then $L_1 = 0$, then $L_2 = 1$, then $\frac{\partial F}{\partial z_2} = 1$ contrary to assumption.
- So, $K_2 < 1$,

$$\frac{1 - z_2\bar{w}_2}{1 - z_1\bar{w}_2} = \frac{K_1}{1 - K_2} = K_1 \sum_{j=0}^{\infty} K_2^j$$

Pick's theorem $\implies$ z_2 is a function of z_1.