Elementary geometry from an advanced point of view

Math 302 — Fall 2006

Postulates and theorems

Postulate I-0. All lines and planes are sets of points.

Postulate I-1. Given any two different points, there is exactly one line containing them.

Postulate I-2. Given any three different noncollinear points, there is exactly one plane containing them.

Postulate I-3. If two points lie in a plane, then the line containing them lies in the plane.

Postulate I-4. If two planes intersect, then their intersection is a line.

Postulate I-5. Every line contains at least two points. S contains at least three noncollinear points. Every plane contains at least three noncollinear points. And S contains at least four noncoplanar points.

Theorem 2.1. *Two different lines intersect in at most one point.*

Theorem 2.2. *If a line intersects a plane not containing it, then the intersection is a single point.*

Theorem 2.3. *Given a line and a point not on the line, there is exactly one plane containing them.*

Theorem 2.4. *If two lines intersect, then their union lies in exactly one plane.*

Postulate D-0. d is a function $d\colon S \times S \to \mathbf{R}$.

Postulate D-1. For every $P, Q \in S$, $d(P,Q) \geq 0$.

Postulate D-2. $d(P,Q) = 0$ if and only if $P = Q$.

Postulate D-3. $d(P,Q) = d(Q,P)$ for all $P, Q \in S$

Postulate D-4 (Ruler postulate). Every line has a coordinate system.

Remark. Note that D-4 implies D-1, D-2 and D-3.

Theorem 3.3.1. *If f is a coordinate system for L, and $g(P) = -f(P)$ for each $P \in L$, then g is a coordinate system for L.*

Theorem 3.3.2. *Let f be a coordinate system for the line L. Let a be any real number* *and for each* $P \in L$*, let* $g(P) = f(P) + a$*. Then* $g\colon L \to \mathbf{R}$ *is a coordinate system for* L*.*

Theorem 3.3.3 (The ruler placement theorem). *Let* L *be a line, and let* P *and* Q *be any two points of* L*. Then* L *has a coordinate system in which the coordinate of* P *is 0 and the coordinate of* Q *is positive.*

Theorem B-1. *If* A*-*B*-*C*, then* C*-*B*-*A*.*

Theorem B-2. *Of any three points on a line, exactly one is between the other two.*

Theorem B-3. *Any four points of a line can be named in an order* A*,* B*,* C*,* D *in such a way that* A*-*B*-*C*-*D*.*

Theorem B-4. *If* A *and* B *are any two points, then (1) there is a point* C *such that* A*-*B*-*C*, and (2) there is a point* D *such that* A*-*D*-*B*.*

Theorem B-5. *If* A*-*B*-*C*, then* A*,* B *and* C *are three different points on the same line.*

Theorem 3.5.1. *If* A *and* B *are any two points, then* $\overline{AB} = \overline{BA}$*.*

Theorem 3.5.2. *If* C *is a point of* $\overrightarrow{AB}$ *other than* A*, then* $\overrightarrow{AB} = \overrightarrow{AC}$*.*

Theorem 3.5.3. *If* B' *and* C' *are points of* $\overrightarrow{AB}$ *and* $\overrightarrow{AC}$ *other than* A*, then* $\angle BAC = \angle B'AC'$*.*

Theorem 3.5.4. *If* $\overline{AB} = \overline{CD}$*, then the points* A, B *are the same as the points* C, D *in some order.*

Theorem 3.5.5. *If* $\triangle ABC = \triangle DEF$*, then the points* A, B, C *are the same as the points* D, E, F *in some order.*

Theorem C-1. *For segments, congruence is an equivalence relation.*

Theorem C-2 (The segment-construction theorem). *Given a segment* $\overline{AB}$ *and a ray* $\overrightarrow{CD}$*. There is exactly one point* E *of* $\overrightarrow{CD}$ *such that* $\overline{AB} \cong \overline{CD}$*.*

Theorem C-3 (The segment-addition theorem). *If* A*-*B*-*C*,* A'*-*B'*-*C'*,* $\overline{AB} \cong \overline{A'B'}$ *and* $\overline{BC} \cong \overline{B'C'}$*, then* $\overline{AC} \cong \overline{A'C'}$*.*

Theorem C-4 (The segment-subtraction theorem). *If* A*-*B*-*C*,* A'*-*B'*-*C'*,* $\overline{AB} \cong \overline{A'B'}$ *and* $\overline{AC} \cong \overline{A'C'}$*, then* $\overline{BC} \cong \overline{B'C'}$*.*

Theorem C-5. *Every segment has exactly one midpoint.*

Postulate PS-1 (The plane-separation postulate). Given a line and a plane containing it, the set of all points of the plane that do not lie on the line is the union of two disjoint sets such that

(1) each of the sets is convex, and

(2) if P belongs to one of the sets and Q belongs to the other, then the segment $\overline{PQ}$ intersects the line.

Theorem (The postulate of Pasch). *Given a triangle* $\triangle ABC$*, and a line* L *in the same plane. If* L *contains a point* P*, between* A *and* C*, then* L *intersects at least one of* $\overline{AB}$ *and* $\overline{BC}$*.*

Theorem 4.2.1. *If P and Q are on opposite sides of the line L, and Q and T are on opposite sides of L, then P and T are on the same side of L.*

Theorem 4.2.2. *If P and Q are on opposite sides of the line L, and Q and T are on the same side of L, then P and T are on opposite sides of L.*

Theorem 4.2.3. *Given a line, and a ray which has its endpoint on the line, but does not lie on the line. Then all points of the ray, except for the endpoint, are on the same side of the line.*

Theorem 4.2.4. *Let L be a line, let A and B be points with $A \in L$ and $B \notin L$. Then all points of $\overline{AB} \setminus \{A\}$ lie on the same side of L.*

Theorem 4.2.5. *Every side of a triangle lies, except for its endpoints, in the interior of the opposite angle.*

Theorem 4.2.6. *If F is in the interior of $\angle BAC$, then $\overrightarrow{AF} \setminus \{A\}$ lies in the interior of $\angle BAC$.*

Theorem 4.1.8. *If A and B are convex, then so is also $A \cap B$.*

Theorem 4.1.9. *If G is any collection of convex sets G_i, then $\bigcap_{G_i \in G} G_i$ is convex.*

Theorem 4.1.10. *If A is any set of points, then the convex hull of A is convex.*

Theorem 4.2.8. *The interior of a triangle is always a convex set.*

Theorem 4.2.9. *The interior of a triangle is the intersection of the interiors of its angles.*

Theorem 4.3.1. *Let L be a line, let A and F be two points of L and let B and G be points on opposite sides of L. Then $\overline{FB}$ does not intersect $\overrightarrow{AG}$.*

Theorem 4.3.2. *In $\triangle FBC$, let A be a point between F and C, and let D be a point such that D and B are on the same side of $\overleftrightarrow{FC}$. Then $\overrightarrow{AD}$ intersects at least one of $\overline{FB}$ and $\overline{BC}$.*

Theorem 4.3.3 (The crossbar theorem). *If D is in the interior of $\angle BAC$, then $\overrightarrow{AD}$ intersects $\overline{BC}$, in a point between B and C.*

Theorem 4.4.1. *The diagonals of a convex quadrilateral always intersect each other.*

Theorem 4.4.1'. *If the diagonals of a quadrilateral intersect each other, then the quadrilateral is convex.*

Postulate SS-1 (The space-separation postulate). Given a plane in space. The set of all points that do not lie in the plane is the union of two disjoint sets such that

(1) each of the sets is convex, and

(2) if P belongs to one of the sets and Q belongs to the other, then the segment $\overline{PQ}$ intersects the plane.

Postulate M-1. m is a function $\mathcal{A} \to \mathbf{R}$.

Postulate M-2. For every angle $\angle A$, $m\angle A$ is between 0 and 180.

Postulate M-3 (The angle-construction postulate). Let $\overrightarrow{AB}$ be a ray on the edge of the half plane H. For every number r between 0 and 180, there is exactly one ray $\overrightarrow{AP}$, with P in H, such that $m\angle PAB = r$.

Postulate M-4 (The angle-addition postulate). If D is in the interior of $\angle BAC$, then $m\angle BAC = m\angle BAD + m\angle DAC$.

Postulate M-5 (The supplement postulate). If two angles form a linear pair, then they are supplementary.

Theorem 5.1. *For angles, congruence is an equivalence relation.*

Theorem 5.2 (The angle-construction theorem). *Let $\angle ABC$ be an angle, let $\overrightarrow{B'C'}$ be a ray and let H be a half plane whose edge contains $\overrightarrow{B'C'}$. Then there is exactly one ray $\overrightarrow{B'A'}$ with A' in H such that $\angle ABC \cong \angle A'B'C'$.*

Theorem 5.3 (The angle-addition theorem). *If D is in the interior of $\angle BAC$, D' is in the interior of $\angle B'A'C'$, $\angle BAD \cong \angle B'A'D'$ and $\angle DAC \cong \angle D'A'C'$, then $\angle BAC \cong \angle B'A'C'$.*

Theorem 5.4 (The angle-subtraction theorem). *If D is in the interior of $\angle BAC$, D' is in the interior of $\angle B'A'C'$, $\angle BAD \cong \angle B'A'D'$ and $\angle BAC \cong \angle B'A'C'$, then $\angle DAC \cong \angle D'A'C'$.*

Theorem 5.5 (The vertical angle theorem). *If two angles form a vertical pair, then they are congruent.*

Theorem 5.6. *If two intersecting lines form one right angle, then they form four right angles.*

Postulate SAS. Given a correspondence between two triangles (or between a triangle and itself). If two sides and the included angle of the first triangle are congruent to the corresponding parts of the second triangle, then the correspondence is a congruence.

Theorem 6.2.1 (The isosceles triangle theorem). *If two sides of a triangle are congruent, then the angles opposite them are congruent.*

Theorem 6.2.2 (ASA). *Given $\triangle ABC$ and $\triangle DEF$. If $\angle A \cong \angle D$, $\overline{AC} \cong \overline{DF}$ and $\angle C \cong \angle F$, then $\triangle ABC \cong \triangle DEF$.*

Theorem 6.2.3 (SSS). *Given $\triangle ABC$ and $\triangle DEF$. If $\overline{AB} \cong \overline{DE}$, $\overline{AC} \cong \overline{DF}$ and $\overline{BC} \cong \overline{EF}$, then $\triangle ABC \cong \triangle DEF$.*

Theorem 6.2.4. *Every angle has exactly one bisector.*

Theorem 7.7 (SAA). *Given $\triangle ABC$ and $\triangle DEF$. If $\overline{AB} \cong \overline{DE}$, $\angle B \cong \angle E$, and $\angle C \cong \angle F$, then $\triangle ABC \cong \triangle DEF$.*

Theorem 7.1. *Any exterior angle of a triangle is greater than each of its remote interior angles.*

Theorem 6.5.1/8.3.4. *Given a line and a point, then there is a unique line which passes through the given point and is perpendicular to the given line.*

Theorem 10.1.1. *If two lines are perpendicular to the same line, then the two lines are* *parallel.*

Theorem 10.1.2. *If P is a point off line L, then there is a line through P that is parallel to L.*

Theorem 7.2. *Given $\triangle ABC$. If $\overline{AB} > \overline{AC}$, then $\angle C > \angle B$.*

Theorem 7.3. *Given $\triangle ABC$. If $\angle C > \angle B$, then $\overline{AB} > \overline{AC}$.*

Theorem 7.4. *The shortest segment joining a point to a line is the perpendicular segment.*

Theorem 7.5 (The triangular inequality). *If A, B, C are noncollinear, then $AB + BC > AC$.*

Theorem 10.2.1. *For any points A, B, C, $AB + BC \geq AC$.*

Theorem 7.6 (The hinge theorem). *Given $\triangle ABC$ and $\triangle DEF$. If $\overline{AB} \cong \overline{DE}$, $\overline{AC} \cong \overline{DF}$ and $\angle A > \angle D$, then $\overline{BC} > \overline{EF}$.*

Theorem 7.8 (The hypotenuse-leg theorem). *Given triangles $\triangle ABC$ and $\triangle DEF$ with $m\angle A = m\angle D = 90$. If $\overline{AB} \cong \overline{DE}$ and $\overline{BC} \cong \overline{EF}$, then $\triangle ABC \cong \triangle DEF$.*

Theorem 10.1.3. *Given two lines and a transversal. If a pair of alternate interior angles are congruent, then the lines are parallel.*

Theorem 10.1.4. *Given two lines and a transversal. If a pair of corresponding angles are congruent, then a pair of alternate interior angles are congruent.*

Theorem 10.1.5. *Given two lines and a transversal. If a pair of corresponding angles are congruent, then the lines are parallel.*

Theorem 10.3.1. *The diagonals of a Saccheri quadrilateral are congruent.*

Theorem 10.3.2. *Let $\square ABCD$ and $\square A'B'C'D$ be Saccheri quadrilaterals with lower bases $\overline{AD}$ and $\overline{A'D'}$. If $\overline{AD} \cong \overline{A'D'}$ and $\overline{AB} \cong \overline{A'B'}$, then $\overline{BC} \cong \overline{B'C'}$, $\angle B \cong \angle B'$ and $\angle C \cong \angle C'$.*

Theorem 10.3.3. *In any Saccheri quadrilateral, the upper base angles are congruent.*

Theorem 10.3.4. *In any Saccheri quadrilateral, the upper base is congruent to or longer than the lower base.*

Theorem 10.2.2 (The polygonal inequality). *If $A_1, A_2, \ldots, A_n$ are any points ($n \geq 2$), then $A_1A_2 + A_2A_3 + \cdots + A_{n-1}A_n \geq A_1A_n$.*

Theorem 10.4.1. *In any Saccheri quadrilateral $\square ABCD$ with lower base $\overline{AD}$, we have $\angle BDC \geq ABD$.*

Theorem 10.4.2. *If $\triangle ABD$ has a right angle at A, then $m\angle B + m\angle D \leq 90$.*

Theorem 10.4.3. *Every right triangle has only one right angle, and its other two angles are acute.*

Theorem 10.4.4. *The hypotenuse of a right triangle is longer than either of the legs.*

Theorem 10.4.5. *In $\triangle ABC$, let D be the foot of the perpendicular from B to $\overleftrightarrow{AC}$. If $\overline{AC}$ is the longest side of $\triangle ABC$, then A-D-C.*

Theorem 10.4.6. *In any triangle $\triangle ABC$, we have $m\angle A + m\angle B + m\angle C \leq 180$.*

Postulate P-1 (Euclidean parallel postulate). Given a line and an external point, there is only one line which passes through the given point and is parallel to the given line.

Theorem 11.1.1. *Given two lines and a transversal. If the lines are parallel, then each pair of alternate interior angles is congruent.*

Theorem 11.1.2. *Given two lines and a transversal. If the lines are parallel, then each pair of corresponding angles is congruent.*

Theorem 11.1.3. *In any triangle* $\triangle ABC$ *we have* $m\angle A + m\angle B + m\angle C = 180$.

Theorem 11.1.4. *The acute angles of a right triangle are complementary.*

Theorem 11.1.5. *Every Saccheri quadrilateral is a rectangle.*

Theorem 11.1.6. *For any triangle, the measure of an exterior angle is the sum of the measures of its two remote interior angles.*

Theorem 11.1.7. *In a plane, any two lines parallel to a third line are parallel to each other.*

Theorem 11.1.8. *If a transversal is perpendicular to one of two parallel lines, it is perpendicular to the other.*

Theorem 11.1.9. *Either diagonal divides a parallelogram into two congruent triangles.*

Theorem 11.1.10. *In a parallelogram, each pair of opposite sides are congruent.*

Theorem 11.2.1. *Every parallel projection is a one-to-one correspondence.*

Theorem 11.2.2. *Parallel projections preserve betweenness.*

Theorem 11.2.3. *Parallel projections preserve congruence.*

Theorem 11.4.1 (The basic similarity theorem). *Let* L_1, L_2, *and* L_3 *be three parallel lines, with common transversals* T *and* T' *intersecting them in points* A, B, C *and* A', B', C'. *If* A-B-C *(and* A'-B'-C'*), then* $\frac{BC}{AB} = \frac{B'C'}{A'B'}$.

Theorem 11.4.2. *If two segements on the same line have no points in common, then the ratio of their lengths is preserved under every parallel projection.*

Theorem 11.4.3. *Parallel projections preserve ratios.*