Homework 10: Due 11/30/2017

1. Problem 2 on page 217 of Shao (2003).
2. Problem 4 on page 217 of Shao (2003).
3. Problem 6 on page 218 of Shao (2003).
4. Problem 9 on page 218 of Shao (2003).
5. Jackknife estimator. Let $T(\boldsymbol{X})$ be an estimator of $g(\theta)$ based on a random sample $\boldsymbol{X} = (X_1, \cdots, X_n)$. Suppose

$$E(T(\boldsymbol{X})) = g(\theta) + \sum_{k=1}^{\infty} \frac{a_k}{n^k},$$

where a_k may depend on θ, but not on n. Define $\boldsymbol{X}_{(-i)} := (X_1, \cdots, X_{i-1}, X_{i+1}, \cdots, X_n)$ to be the vector of sample after deleting the i-th observation. Consider

$$T_J(\boldsymbol{X}) = nT(\boldsymbol{X}) - \frac{n-1}{n} \sum_{i=1}^{n} T(\boldsymbol{X}_{(-i)}),$$

which is referred to as the first-order jackknife estimator.

   (a) Show that the expectation of the jackknife estimator, T_J is

   $$E(T_J(\boldsymbol{X})) = g(\theta) - \frac{a^2}{n^2} + O(n^{-3}).$$

   (b) Show that if the order of $Var(T)$ is n^{-1}, then the order of $Var(T_J)$ is also n^{-1}. Remark: Hence the jackknife estimator can reduce bias (eliminate the first order bias) but not increase the variance.

   (c) To estimate variance of T, one may consider

   $$U_n = \frac{1}{n} \sum_{i=1}^{n} (T(\boldsymbol{X}_{(-i)}) - T(\boldsymbol{X}))^2,$$

   which is referred to as jackknife variance estimator. Show that it can be viewed as U-statistic by specifying the kernel and order. In particular, if $T = \bar{X}$, show that $U_n = S_n^2/n$, where S_n^2 is the sample variance.

6. Show the corollary of Hoeffding's theorem for U-statistics. Assume $E(h^2) < \infty$.

   (a) $\xi_1 \leq \xi_2 \leq \cdots \xi_m$

   (b) $\frac{m^2}{n}\xi_1 \leq Var(U_n) \leq \frac{m}{n}\xi_m$

   (c) For any $n > m$, $(n+1)Var(U_{n+1}) \leq nVar(U_n)$.

   (d) For a fixed m, let $k_0 = \min_k\{k : \xi_k > 0\}$. Then $Var(U_n) = \frac{k!\binom{m}{k}^2 \xi_k}{n^k} + O(\frac{1}{n^{k+1}})$. Therefore, U-statistic has MSE of the order n^{-k} and is $n^{k/2}$-consistent.