Homework 3: Due 09/21/2017

1. Uniform distribution.

   (a) Consider an arbitrary random variable X with c.d.f F. Show that $Y = F(X) \sim U(0, 1)$.

   (b) Consider a random variable $U \sim U(0, 1)$ and an arbitrary c.d.f F. Show that $Y = F^{-1}(U)$ is a random variable and has c.d.f F. (Here $F^{-1}(t) = \inf\{x : F(x) \geq t\}$.)

   Remark: The first result can be used to show p-value in most of statistical tests is a $U(0, 1)$ random variable. The second result are often used to generate random variable with known distribution function.

2. Problem 43 + 65 + 81(a) on page 78 of Shao (2003).
   Properties of Normal.

3. Show Cochran's Theorem.
   Let $X \sim N(0, I_n)$ and $Q_i = X^T A_i X$ for a nonnegative definite $n \times n$ matrix A_i with $rank(A_i) = r_i$, $i = 1, \cdots, k$. Suppose $X^T X = \sum_{i=1}^{k} Q_i$.

   (a) Prove that $Q_i \sim \chi_r^2$ iff A_i has r eigenvalues being 1 and all the other eigenvalues 0. (A_i is idempotent.)

   (b) Prove that $Q_i \perp\!\!\!\perp Q_j$ iff $A_i A_j = 0$.

   (c) Prove if $Q_i \sim \chi_{r_i}^2$, for $i = 1, \cdots, k$, then Q_i are independent and $\sum_{i=1}^{k} r_i = n$.

   (d) Prove if $Q_i, i = 1, \cdots, k$ are independent, then $Q_i \sim \chi_{r_i}^2$ and $\sum_{i=1}^{k} r_i = n$.

4. Let $X \sim N(\mu, \sigma^2 I_k)$, where μ is a constant in $\mathbb{R}^k$. Consider

   $$\bar{X} = \mathbf{1}^T X / k$$

   and

   $$S^2 = (X - \bar{X}\mathbf{1})^T (X - \bar{X}\mathbf{1})/(k - 1).$$

   (d) Show that $(k - 1)S^2/\sigma^2 \sim \chi_{k-1}^2$.
   (e) Show that $\bar{X}$ and S^2 are independent.
   (f) Show that $T_0 = (\bar{X} - \mu)/\sqrt{S^2/k} \sim T_{k-1}$.
   (g) Let $\delta = \mu - \mu_0$ for some hypothetical μ_0 and $T_1 = (\bar{X} - \mu_0)/\sqrt{S^2/k}$. Derive the distribution of T_1.

5. Problem 78 on page 82 of Shao (2003).
   Distribution of sum of independent random variables.