Homework 4: Due 09/28/2017

1. Let $(\Omega, \mathcal{F}, P)$ be the unit interval $[0,1]$ with Lebesgue measure P on the Borel σ-field restricted on $[0,1]$. Let X be r.v. on $(\Omega, \mathcal{F}, P)$. Consider $\mathcal{G} = \sigma(\{G : G \in \Omega, G \text{ is countable or } G^c \text{ is countable}\})$.

   (a) What is the conditional probability, $P(A|\mathcal{G}), A \in \mathcal{F}$?

   (b) What is $E(X|\mathcal{G})$?

   (c) Prove that X is independent of $\mathcal{G}$.

2. Consider the example 1.23 on page 44 of Shao (2003).
   (a) If the number of customers who respond follows Poisson distribution with parameter θ ($EY = \theta$), what is the distribution of X? (This is the case if one consider infinite sample size N and fixed response rate $\theta = N\pi$.)
   (b) Assume each customer behave independently and has same tendency (probability) π to respond. If one stops the survey when the number of non-response reaches $r, r \in \mathbb{N}$, then what is the distribution of Y and what is the distribution of X?

3. (a) Let $\mathcal{G}_1 \subset \mathcal{G}_2$ and $EX^2 < \infty$. Show that

$$E\left[X - E(X|\mathcal{G}_2)\right]^2 \leq E\left[X - E(X|\mathcal{G}_1)\right]^2.$$

   The dispersion of X about its conditional mean becomes smaller as σ-field grows.
   (b) As a special case of part (a), if $\mathcal{G}_2 = \sigma(Y)$ and $\mathcal{G}_1 = \sigma(Z)$, where $Z = h(Y)$ for some measurable function h, then show that

$$E(Var(X|Y)) \leq E(Var(X|Z)).$$

   (b) Prove that

$$Var(X) = E[Var(X|Y)] + Var[E(X|Y)].$$

4. Problem 82 on page 80 of Shao (2003).
   Let $(X_1, X_2)^T$ be a multivariate normal random vector.,

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim N(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix})$$

   Derive the conditional distribution of X_1 given X_2.