Homework 5: Due 10/05/2017

1. Let X, Y be independent standard normal random variables, and (R, Θ) be the polar coordinates for (X, Y):

$$R = \sqrt{X^2 + Y^2}, \quad \Theta = \arctan 2(Y, X),$$

where

$$\arctan 2(y, x) = \begin{cases} \arctan(y/x), & x > 0 \\ \arctan(y/x) + \pi, & x < 0, y \geq 0 \\ \arctan(y/x) - \pi, & x < 0, y < 0 \\ \pi/2, & x = 0, y > 0 \\ -\pi/2, & x = 0, y < 0. \end{cases}$$

   (a) Show that $R \perp\!\!\!\perp \Theta$, and R^2 given Θ follows χ^2_2.

   (b) Show that R^2 given $X - Y$ follows χ^2_1 translated by $(X - Y)^2/2$.
   (Hint: $R^2 = [(X + Y)^2 + (X - Y)^2]/2$ and $X + Y$ is independent of $X - Y$.)

   (c) From part (b), given $X - Y = 0$, R^2 follows χ^2_1. And from part (a), given $\Theta = \pi/4$ or $\Theta = 5\pi/4$, R^2 follows χ^2_2. But $\{X - Y = 0\}$ and $\{\Theta = \pi/4\} \cup \{\Theta = 5\pi/4\}$ are the same. Explain the paradox. What is the distribution of R^2 given $X = Y$?

2. Poisson process.
Let $\{X_i : i = 1, 2, \cdots\}$ be a sequence of positive random variables on $(\Omega, \mathcal{F}, P)$. Image X_1 as the waiting time for the occurrence of the first event such as the arrival of customer in a bank, and X_k as the waiting time from the $k - 1$th event to the k-th event. Consider the partial sum

$$Y_k = \sum_{i=1}^{k} X_i,$$

which represents waiting time for the k-th event. Let $N_t = 0$ and for a given $t > 0$, let

$$N_t = \max_k \{k : Y_k \leq t\}.$$

Then $\{N_t : t \geq 0\}$ is called a counting process. Assume no two events are to occur simultaneously ($X_i > 0, \forall i$), and only finitely many of the events are to occur in each finite interval of time ($\lim_n \sum_{i=1}^{n} X_i = \infty$).

   (a) If $X_i \sim E(0, \theta)$ (Exponential distribution), then X_i is memoryless in the sense that

$$P(X_i > s + t | X_i > s) = P(X > t), \quad \forall s, t > 0.$$

   (b) Let Z be a positive continuous random variable with memoryless property. Show that Z follows exponential distribution. (So memoryless waiting time has to be exponential.)

(c) Assume $X_i \stackrel{\text{iid}}{\sim} E(0, \theta)$, for $i = 1, 2, \cdots$. Derive the distribution of N_t for any fixed t.

(d) Assume $\{N_t : t \geq 0\}$ has independent Poisson increment. That is
(i) For any $0 < t_1 < \cdots < t_n$, the increments $N_{t_1}, N_{t_2} - N_{t_1}, \cdots, N_{t_n} - N_{t_{n-1}}$ are independent;
(ii) The individual increment follows Poisson distribution with

$$P(N_t - N_s = m) = \frac{(\theta(t-s))^m e^{-(\theta(t-s))}}{m!}, \quad m = 0, 1, 2,$$

for $0 \leq s < t$. Then this stochastic process $\{N_t : t \geq 0\}$ is called a **Poisson process**. Show that $X_i \stackrel{\text{iid}}{\sim} E(0, \theta^{-1})$.

Remark: Under some regularity conditions, (ii) can be relaxed. Prékopà's Theorem shows that if the independent increment has distribution only depends on $t - s$, then the $\{N_t : t \geq 0\}$ has to be a Poisson process too.

(e) Show that a Poisson process $\{N_t : t \geq 0\}$ is a Markov process (satisfying Markov property).

(f) Let $\{N_t : t \geq 0\}$ be a Poisson process (or $X_i \stackrel{\text{iid}}{\sim} E(0, \theta^{-1})$). Prove that the conditional distribution N_s given N_t for $0 < s < t$ is $Bin(s/t, N_t)$.

3. Problem 106 on Page 85 Shao 2003.
   Show martingales.

4. Problem 116 on Page 86 Shao 2003.
   Examples to show the relationship between different convergence modes.