Homework 6: Due 10/19/2017

1. Problem 121 on Page 85 Shao 2003.
   Convergence in probability together with uniformly bounded implies convergence in moment.

2. The closure of a set A is defined as $\bar{A} = A \cup \{\lim_n a_n : a_n \in A\}$. If $\bar{A} = \mathbb{R}$, we say the set A dense in $\mathbb{R}$.

   (a) Prove that if a sequence of c.d.f. $\{F_n\}$ converges to a c.d.f F for all points in a dense set, then $F_n \xrightarrow{w} F$.

   (b) Let F and G be two c.d.f.s. Prove, if $\lim_{n\to\infty} F_n(x) = F(x)$ and $\lim_{n\to\infty} F_n(x) = G(x)$ for all common continuous points x of F and G, then $F = G$. In another word, the limiting c.d.f. is unique.

   (c) In statistics, we often need to show the limiting distribution of $Y_n = \frac{X_n - b_n}{a_n}$, for some $a_n > 0$ and $a_n, b_n \in \mathbb{R}$. Note the c.d.f. of Y_n can be writen as $G_n(t) = F_{X_n}(a_n t + b_n)$. Suppose $Y_n \xrightarrow{D} Y$ and $X_n \xrightarrow{D} X$ with nondegenerate c.d.f. G and F, respectively. Show that $a_n \to \sigma > 0$, $b_n/a_n \to \mu \in \mathbb{R}$, and $G(t) = F(\sigma t + \mu)$, for all $t \in \mathbb{R}$. (We say F and G are **of the same type**.)
   Remark: As a special case of part (c), let $X_n \sim N(\mu_n, \sigma_n^2)$ and $X \sim N(\mu, \sigma^2)$. Then $X_n \xrightarrow{D} X$ iff $\mu_n \to \mu$ and $\sigma_n \to \sigma$.

3. Let $\{X_n\}$ be a sequence of random variables.

   (a) If $E|X_n| = O(a_n)$, for some nonnegative $a_n \in \mathbb{R}$, then $X = O_p(a_n)$.

   (b) If $X_n \xrightarrow{a.s} X$, then $\sup_n |X_n| = O_p(1)$.

4. Let $U_i \overset{\text{iid}}{\sim} U[0,1]$.

   (a) Let $Y_n = (\Pi_{i=1}^n U_i)^{-1/n}$. Show that $\sqrt{n}(Y_n - e) \xrightarrow{D} N(0, e^2)$.

   (b) Let $Z_n = n(1 - U_{(n-1)})$, where $U_{(k)}$ is the kth smallest U_i (kth order statistics) for $k = 1, \cdots, n$. Find the distribution of Z_n.

5. Let $X_n \sim Bin(p_n, n), 0 < p_n < 1$.

   (a) If $E(X_n) = np_n \to \theta > 0$, derive the limiting distribution of X_n.

   (b) If $Y_n = X_n/n$ and $p_n \equiv p$, prove that $Y_n \xrightarrow{p} p$.

   (c) Note that $E(X_n) = np_n$ and $Var(X_n) = np_n(1 - p_n)$. Given $p_n \to p$, prove that $(X_n - np)/\sqrt{np(1-p)} \xrightarrow{D} N(0,1)$.