Homework 7: Due 11/02/2017

1. Let $Y_i = 1/X_i^2, i = 1, \cdots, n$, where $X_i \stackrel{\text{iid}}{\sim} N(0,1)$.
   (a) Derive the pdf of Y_1.
   (b) Show that the limiting distribution of $\bar{Y}_n = \sum_{i=1}^n Y_i/n$ is same as the distribution of nY_1.

2. Problem 161 on page 90 of Shao (2003)

3. Let $X_i \sim N(\mu, \sigma^2)$ and consider the estimation of μ^2. The maximum likelihood estimator for μ^2 is $T_{1n} = \bar{X}^2$, where $\bar{X} = \sum_{i=1}^n X_i/n$. However, we know $E(\bar{X}^2) > \mu^2$, so another reasonable (unbiased) estimator for μ^2 is $T_{2n} = \bar{X}^2 - S^2/n$, where $S^2 = \sum_{i=1}^n (X_i - \bar{X})^2/(n-1)$. Obtain the limiting distribution of
   (a) $\sqrt{n}(T_{1n} - \mu^2)$, for $\mu \neq 0$.
   (b) $\sqrt{n}(T_{2n} - \mu^2)$, for $\mu \neq 0$.
   (c) $a_n(T_{1n} - \mu^2)$, where $\mu = 0$ and a_n is a suitable sequence of real numbers such that $a_n(T_{1n} - \mu^2)$ has nondegenerate limiting distribution.
   (d) $b_n(T_{2n} - \mu^2)$, where $\mu = 0$ and b_n is a suitable sequence of real numbers such that $b_n(T_{2n} - \mu^2)$ has nondegenerate limiting distribution.

4. Let $X_1, \cdots, X_n$ be iid random variables with cdf $F(x;\theta)$, where $F(x;\theta) = F(x - \theta)$ (location family) and $F(0) = 1/2$. Hence the median of the distribution is θ. It is natural to consider the sample median, $\tilde{X}_n$ to estimate θ. Assume F is differentiable and $F'(0) = f(0) > 0$. Let $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$ be the ordered sample. Define

$$\tilde{X}_n = \begin{cases} X_{(m)}, & n = 2m-1 \\ \left(X_{(m)} + X_{(m+1)}\right)/2, & n = 2m, \end{cases}$$

   for some $m \in \mathbb{N}$. Please investigate the limiting behavior of $\sqrt{n}(\tilde{X}_n - \theta)$.
   (a) First consider odd sample size, i.e. $n = 2m-1$ and $\tilde{X}_n = X_{(m)}$. Show that

$$\sqrt{n}(\tilde{X}_n - \theta) \xrightarrow{D} N(0, 1/(4f^2(0))).$$

   (Hint: first show $P(\sqrt{n}(\tilde{X}_n - \theta) \leq t) = P(Y_n \leq (n-1)/2)$ where $Y_n = \sum_{i=1}^n \mathbb{1}\{X_i > \theta + t/\sqrt{n}\}$. Note here $Y_n \sim Bin(p_n, n)$ where $p_n = 1 - F(t/\sqrt{n})$.)
   (b) Then consider even sample size. Show the previous result still holds.
   (Hint: first show the result still holds if $\tilde{X}_n$ is replaced by either $X_{(m)}$ or $X_{(m+1)}$. Note $X_{(m)} \leq \tilde{X}_n \leq X_{(m+1)}$.)

5. Lindeberg's and Feller's conditions. (The results are more important in practice.)

   (a) Show that the Liapounov's condition (moment conditions) implies the Lindeberg's condition.
   Problem 157 on page 90 of Shao (2003).

   (b) Show that the Feller's condition implies uniform asymptotic negligible, i.e. for $\forall \epsilon > 0$,

   $$\lim_{n\to\infty} \max_{1\le j\le k_n} P(|X_{nj}^*)| > \epsilon) = 0,$$

   where X_{nj}^* is the standardized X_{nj}, i.e. $X_{nj}^* = (X_{nj} - E(X_{nj}))/\sigma_n$

   (c) Consider X_i independently follows $N(0, 1/2^i), i = 1, 2, \cdots, n$. Show that both the Lindeberg's condition and the Feller's condition fail, and

   $$\frac{1}{\sigma_n}\sum_{i=1}^{n}(X_i - E(X_i)) \xrightarrow{D} N(0,1).$$

   (d) Consider linear regression model $Y = X\beta + e$, with $E(e) = 0, Var(e) = \sigma^2 I_n$, where Y and e are $n\times 1$ vector, X is a $n \times p$ matrix, and β is $p\times 1$ vector. The solution that minimizes $\|Y - X\beta\|^2$ is called the **least square estimator**(LSE) for β. It can be shown that

   $$\hat{\beta} = (X^TX)^{-1}X^TY$$

   is the LSE. And $E(\hat{\beta}) = \beta$ and $Var(\beta) = \sigma^2(X^TX)^{-1}$. Then the standardized LSE

   $$(X^TX)^{1/2}(\hat{\beta} - \beta) = \sum_{i=1}^{n} a_{ni}e_i,$$

   where $a_{n1}, \cdots, a_{nn}$ are the columns of the $p \times n$ matrix $(X^TX)^{-1/2}X^T$. State the Lindeberg condition in terms of a_{ni} and e_i such that

   $$(X^TX)^{1/2}(\hat{\beta} - \beta) \xrightarrow{D} N(0,1).$$

   Remark: this result does not need normality assumption on e_i, but only independence.