Homework 12, Math 4111, due 5 Dec 2013

Do not submit problems in blue, but at least attempt them.

(1) Let α be the function on $[a, b]$ defined as $\alpha(x) = 0$ if x is rational and $\alpha(x) = 1$ if x is irrational. Prove that if $f \in R(\alpha)$, then f is constant. (Hint: It may be easier to use $\alpha \in R(f)$, using integration by parts)

(2) (a) Let $P(x, y)$ be a continuous function on $[0, 1] \times [0, 1]$ which has continuous derivative in x. Let $F(x) = \int_0^x P(x, y)dy$ for $x \in [0, 1]$. Prove that F is differentiable and $F'(x) = P(x, x) + \int_0^x \frac{\partial P(x,y)}{\partial x} dy$. (Caution: x appears as both in the limit of integration and in the integrand in the definition of F)

(b) Let f be continuous on $[0, 1]$ and define $f_{n+1}(x) = \frac{1}{n!} \int_0^x (x - t)^n f(t)dt$ for $x \in [0, 1]$. Prove that f_{n+1} is differentiable and $f'_{n+1} = f_n$ for all integers $n \geq 0$.

(3) Define $f(x) = (\int_0^x e^{-t^2} dt)^2$ and $g(x) = \int_0^1 \frac{e^{-x^2(t^2+1)}}{t^2+1} dt$. Prove that $f'(x) + g'(x) = 0$ for all x and thus $f(x) + g(x) = \pi/4$ for all x. Deduce that $\lim_{x\to\infty} \int_0^x e^{-t^2} dt = \sqrt{\pi}/2$. (Hint: You may use what you have done in calculus. In particular, you may need to change variables using trigonometric functions.)

(4) Here is an alternate definition of Riemann Integral, which may be familiar from Calculus class. Let f be a continuous function on $[a, b]$ and let $n \in \mathbb{N}$. Take a partition of $[a, b]$ by subdividing it into n equal parts. That is, take, $a = x_0 < x_1 < \cdots < x_n = b$ where $x_k = a + \frac{k(b-a)}{n}$. Then $\int_a^b f dx = \lim_{n\to\infty} \frac{b-a}{n} \sum_{k=1}^n f(x_k)$. Prove the above.

(5) Prove that $\lim_{n\to\infty} \sum_{k=1}^n \frac{n}{k^2+n^2} = \frac{\pi}{4}$. (Hint: Same as for problem 3 and the blue part.)

(6) Let f be a positive continuous bounded function on $[a, b]$ which is Riemann integrable and let $M = \sup\{f(x) | x \in [a, b]\}$. Prove that,

$$\lim_{n\to\infty} \left(\int_a^b f(x)^n dx \right)^{1/n} = M.$$