Homework 2, Math 4111, due September 12

(1) Let $x_n = \sum_{k=1}^{n} \frac{1}{k(k+1)}$. Prove that $\lim x_n = 1$. (Hint: $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$.)

(2) Let $x_n = \sum_{k=1}^{n} \frac{1}{k^2}$. Prove that $\lim x_n$ exists. (Hint: The set $\{x_n\}$ is bounded above since $\frac{1}{k^2} \leq \frac{1}{k(k-1)}$ for $k \geq 2$ and the previous problem.)

(3) Assume $\lim x_n = a, \lim y_n = b$. Prove that $\lim x_n + y_n = a + b$ and $\lim x_n y_n = ab$. (Hint: $x_n y_n - ab = (x_n - a)y_n + a(y_n - b)$ and both $\{x_n\}, \{y_n\}$ are bounded.)

(4) Let $x_n = \sum_{k=1}^{n} \frac{1}{k}$. Prove that the set $\{x_n\}$ is not bounded above. (Hint: $\sum_{k=2^m+1}^{2^{m+1}} \frac{1}{k} \geq \frac{1}{2}$.)

(5) Let $\lim x_n = a$ and let $\{y_n\}$ be the sequence, $y_n = \frac{\sum_{k=1}^{n} x_k}{n}$. Prove that $\lim y_n = a$.