Homework 3, Math 4111, due September 19

Do not submit problems in blue, but at least do them.

(1) If A is any set, define $\mathcal{P}(A)$, the *power set of* A to be the set of all subsets of A. Prove that if A is a set with cardinality $n \in \mathbb{N}$, then the power set $\mathcal{P}(A)$ has cardinality 2^n.

(2) Prove that for any set A, $\mathcal{P}(A)$ is bijective to $\mathrm{Mor}(A, \Sigma_2)$, the set of all functions from A to $\Sigma_2 = \{1, 2\}$. (Mor is just a standard notation, Mor being short for *morphisms*.)

(3) Prove that a set A is infinite if and only if there is an injective map from $\mathbb{N}$ to A.

(4) Let $A, B \subset X$ be finite subsets. We will use # to denote cardinality. Prove that $\#(A \cup B) + \#(A \cap B) = \#(A) + \#(B)$.

(5) If A, B are sets with $n, m \in \mathbb{N}$ elements respectively, prove that $A \times B$ has nm elements.

(6) Let A, B be as in the previous problem. Let $\mathrm{Mor}(A, B)$ be the set of all functions from A to B. Prove that this set has m^n elements.

(7) Prove that a set is not equinumerous to its power set. (Hint: If $f : A \to \mathcal{P}(A)$ is a bijection, consider $X \subset A$, defined as $X = \{a \in A | a \notin f(a)\}$.)

(8) Let $f : [0, 1] \to \mathbb{R}$ be a function with the following property. There is a positive constant M such that for any $x_1, x_2, \ldots, x_n \in [0, 1]$ (for any n),

$$|f(x_1) + f(x_2) + \cdots + f(x_n)| \leq M.$$

Prove that the set of points $x \in [0, 1]$ with $f(x) \neq 0$ is countable. (Hint: How many such points are there with $f(x) \geq \frac{1}{k}$ for some $k \in \mathbb{N}$?)