Homework 4, Math 4111, due September 26

(1) Prove that the following examples from class are indeed metric spaces. You only need to verify the triangle inequality.

   (a) Let $\mathcal{C}$ be the set of continuous functions from $[0,1] \to \mathbb{R}$ with the sup norm: $||f|| = \sup_{x\in[0,1]}\{|f(x)|\}$. (You may use any fact that you have studied in Calculus.)

   (b) $\mathbb{Q}$ with the p-adic norm for a prime p. Let me recall it here. If $0 \neq r \in \mathbb{Q}$, we can write it as $p^n(a/b)$ with a, b non-zero integers relatively prime to p and a unique integer n which we call $v_p(r)$, the *valuation* of r. Define $||r|| = 0$ if $r = 0$ and $= p^{-v_p(r)}$ if $r \neq 0$.

   (c) The space ℓ^2 of all sequences $\{x_n\}$ where $x_n \in \mathbb{R}$ with $\sum_{n=1}^{\infty} x_n^2 < \infty$ with the norm $||\{x_n\}|| = \sqrt{\sum x_n^2}$. (Remember that the sum above just means that $\lim_{n\to\infty} \sum_{k=1}^{n} x_k^2$ exists and this limit is the 'infinite' sum). First prove that if $\{x_n\}, \{y_n\} \in \ell^2$ then so is $\{x_n + y_n\}$ and thus the metric $d(\{x_n\}, \{y_n\}) = ||\{x_n - y_n\}||$ is well defined and then it is indeed a metric.

(2) Let d, d' be two metrics on a set M. We say they are *equivalent* if a subset $U \subset M$ is open with respect to the d-metric if and only if it is open with respect to the the d'-metric.

   (a) Prove that d, d' are equivalent if and only if for any $a \in M, r > 0$, there exists an $s > 0$ such that $B_{d'}(a, s) \subset B_d(a, r)$ (open balls of radius s, r with center a in the d', d-metric respectively) and conversely, given $r' > 0$ there exists $s' > 0$ such that $B_d(a, s') \subset B_{d'}(a, r')$.

   (b) Prove that if d is a metric and A is a positive real number, then $d' = Ad$ defined as $d'(a, b) = Ad(a, b)$ is also a metric and equivalent to d.

   (c) Prove that, if d is a metric, then d' defined as,

$$d'(a,b) = \frac{d(a,b)}{1 + d(a,b)}$$

   is also a metric and equivalent to d. (This is often called the *bounded* metric associated to d. Note that $d'(a, b) \leq 1$ for all a, b.)

   (d) Define for $\mathbf{x} = (x_1, \ldots, x_n), \mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$, $d(\mathbf{x}, \mathbf{y}) = |x_1 - y_1| + \cdots + |x_n - y_n|$. Decide whether this is a metric and if so, whether it is equivalent to the Euclidean metric.

(3) Let $f : (M, d) \to (N, d')$ be a function. Prove that the following statements are equivalent.

(a) $f^{-1}(U)$ is open for any $U \subset N$ open.
(b) $f^{-1}(Z)$ is closed for any $Z \subset N$ closed.
(c) Let $f(a) = b$. For any $r > 0$, there exists $s > 0$ such that $f(B_d(a, s)) \subset B_{d'}(b, r)$.
(d) For any subset $S \subset M$ and a an adherent point of S, $f(a)$ is an adherent point of $f(S)$.