Homework 7, Math 4111, due October 24

(1) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous monotonic (increasing or decreasing) function. Prove that f is a homeomorphism to its image $f(\mathbb{R})$. (Recall that f is monotonic increasing if for any $x > y$, $f(x) > f(y)$.) You may prove this assuming increasing, since the decreasing case will be very similar.

(2) Decide whether the continuous functions $f, g : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3 - 3x, g(x) = x^3 + 3x$ are open or not. (Hint: For f, find local maximum or minimum, using calculus and see how these behave near there, but write the proof using only what we have done. For g use the previous problem.)

(3) Let $a, b \in \mathbb{R}$ be such that $a^2 + b^2 = 1$ and let $t = (t_1, t_2) \in \mathbb{R}^2$. Consider the map $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(x, y) = (ax + by + t_1, -bx + ay + t_2)$. Show that there exists a constant $c \in \mathbb{R}$ such that $||f(p) - f(q)|| \leq c||p - q||$ for any $p, q \in \mathbb{R}^2$.

(4) Let $A \subset (M, d)$, a non-empty subset and define $f_A : M \to \mathbb{R}$ by $f_A(x) = d(x, A)$ (Recall that $d(x, A) = \inf\{d(x, a)|a \in A\}$.). Prove that f_A is uniformly continuous.

(5) Let A, B be two disjoint closed subsets of a metric space (M, d). Prove that there exists two disjoint open subsets U, V of M with $A \subset U, B \subset V$. (Hint: Use f_A, f_B)

(6) Let $A, B \subset (M, d)$ be disjoint closed subsets and let $f : X = A \cup B \to (N, d')$ be continuous and further assume that f is uniformly continuous on A, B separately. If A or B is compact, prove that f is uniformly continuous on X.