Homework 9, Math 4111, due November 7

(1) Prove that the map $f : [0,\infty) \to [0,\infty)$ given by $f(x) = x^n$, $n \in \mathbb{N}$ is a homeomorphism. Denoting the inverse as $g(x) = x^{\frac{1}{n}}$, prove that g is differentiable at every point in $(0,\infty)$ and calculate its derivative.

(2) Let $f : (a,b) \to (c,d)$ be a homeomorphism and assume that f is differentiable with finite *non-zero* derivatives at every point in (a,b). Then prove that the inverse $g : (c,d) \to (a,b)$ is differentiable at every point of (c,d).

(3) Recall the division algorithm for polynomials. If P, Q are polynomials, then there exists unique polynomials A, R such that $P = AQ + R$ and $\deg R < \deg Q$. Let $P(x) = x^n + a_1 x^{n-1} + \cdots + a_n$ be a polynomial.

  (a) Prove that if $P(\alpha) = 0$ for some $\alpha \in \mathbb{R}$, then $P = (x-\alpha)Q$ for some polynomial Q (of degree $n-1$).

  (b) Prove that if we also had $P'(\alpha) = 0$, then $Q = (x-\alpha)S$ for some polynomial S.

(4) Let P be an even degree polynomial. Show that we can write P as $\sum c_i Q_i^2$ for some $c_i \in \mathbb{R}$ and Q_i polynomials. (The sum is finite). (Hint: Odd degree polynomials have a root).

(5) Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function such that $|f(a) - f(b)| < M|a-b|^2$ for a fixed positive constant and all $a, b \in \mathbb{R}$. Prove that f is constant.

(6) Assume that $f : \mathbb{R} \to \mathbb{R}$ is a function which has a finite derivative at all points of $\mathbb{R}$. Further assume that $f' : \mathbb{R} \to \mathbb{R}$ is bounded. Prove that f is uniformly continuous on $\mathbb{R}$.

(7) Let $f : [a,b] \to [a,b]$ be a continuous function (from a closed bounded interval to itself) and assume that f is differentiable in (a,b) with $|f'(c)| < 1$ for all $c \in (a,b)$. Prove that there exists a unique $x \in [a,b]$ such that $f(x) = x$.