Homework 1, Math 4121, due 23 Jan 2014

(1) Find the $\limsup$ of the sequence $\{\cos n\}$. (Hint: You may use the fact that the set $\{a + b\pi | a, b \in \mathbb{Z}\}$ is dense in $\mathbb{R}$.)

(2) If $\sum_{n=1}^{\infty} a_n$, where $a_n > 0$, converges, prove that $\sum_{n=1}^{\infty} a_n^2$ converges.

(3) Imitating the proof in class, consider the series $\sum_{n \in S} \frac{1}{n}$, where $S \subset \mathbb{N}$ be the set of all natural numbers which do not have 2 in its decimal representation. Prove that the series converges by showing that it is bounded above by 80.

(4) Let $\{a_n\}$ be a decreasing sequence of positive numbers. Prove that if $\sum_{k=0}^{\infty} 2^k a_{2^k}$ converges, so does $\sum_{n=1}^{\infty} a_n$. (Hint: Use the function $p(n) = 2^n - 1$ to add parenthesis) The converse is also true and the proof is similar.

(5) Decide whether the following series converge 1. $\sum_{n=1}^{\infty} (n^{\frac{1}{n}} - 1)^n$ (Hint: Root test); 2. $\sum_{n=2}^{\infty} \frac{1}{(\log n)^{\log n}}$ (Hint: Compare with the series $\sum \frac{1}{n^2}$.)

(6) Prove that the double series $\sum_{m,n} e^{-(m^2+n^2)}$ converges.