Homework 11, Math 4121, due 17, April 2014

(1) For $0 < r < p < s < \infty$, prove that $L^r \cap L^s \subset L^p$. Further, if $\mu(X) < \infty$, prove that $L^s \subset L^r$ if $0 < r < s < \infty$.

(2) If f, g are positive measurable functions on X with $\mu(X) = 1$ such that $f(x)g(x) \geq 1$ for all $x \in X$, prove that $\int_X f d\mu \int_X g d\mu \geq 1$.

(3) Let $X = (0, \infty)$ and let $f \in C_c(X)$ which is positive. Define $F(x) = \frac{1}{x}\int_0^x f(t)dt$ for $x \in X$. Prove that $F \in L^p(X)$ for any p, $1 < p < \infty$ and $||F||_p \leq \frac{p}{p-1}||f||_p$. (Same is true for any $f \in L^p(X)$.)

(4) Suppose $\mu(X) = 1$ and $f : X \to [0, \infty]$ a measurable function. Let $A = \int_X f d\mu$. Then prove that, $\sqrt{1 + A^2} \leq \int_X \sqrt{1 + f^2} d\mu \leq 1 + A$. If $X = (0, 1)$ with the Lebesgue measure and $f = g'$ for a differentiable function, this must be familiar to you from calculus

(5) Let $p.q$ be conjugate with $1 < p, q < \infty$. For any $f \in L^p$, we have a map $T_f : L^q \to \mathbb{R}$ defined as $T_f(g) = \int_X fg d\mu$.
   (a) If $L : L^q \to \mathbb{R}$ is a linear functional, prove that L is continuous if and only if there exists a non-negative number l such that $|L(g)| \leq l||g||_q$ for all g.
   (b) Prove that $|T_f(g)| \leq ||f||_p||g||_q$. Deduce that T_f is a continuous linear functional.
   (c) Denoting by H the set of all continuous linear functionals on L^q, for any $L \in H$, define $||L||_H = \inf\{l \geq 0 | |L(g)| \leq l||g||_q\}$ and prove that this makes H a normed linear space. (The last phrase just means H is a vector space and the norm has all the basic properties of a norm in L^p-spaces.)
   (d) Prove that the map $A : L^p \to H$ defined by $f \mapsto T_f$ is continuous linear with $||f||_p = ||T_f||_H$ for all f. (In fact, this map is a bijection.)