Homework 12, Math 4121, due 24, April 2014

All problems are routine and should require little or no thinking.

We will denote by Ω_n an open set in $\mathbb{R}^n$.

(1) Let $f : \Omega_n \to \mathbb{R}$ be a function which has all partial derivatives $D_1 f, \ldots, D_n f$ for all points in Ω_n. If $\mathbf{p} \in \Omega_n$ where f has a local maximum or minimum, prove that $D_k f(\mathbf{p}) = 0$ for all k.

(2) Calculate the directional derivatives $f'(\mathbf{p}; \mathbf{u})$ for the following functions $f : \mathbb{R}^n \to \mathbb{R}$ where $\mathbf{u}$ is any vector.
   1) $f(\mathbf{x}) = \mathbf{a} \bullet \mathbf{x}$ for a fixed vector $\mathbf{a}$; 2) $f(\mathbf{x}) = ||\mathbf{x}||^2$.

(3) Let $f : \mathbb{R} \to \mathbb{R}^2$ be defined as $f(x) = (x^2, x^3)$. Given any $\mathbf{a} \in \mathbb{R}^2$, find a z with $0 < z < 1$ such that $\mathbf{a} \bullet (f(1) - f(0)) = \mathbf{a} \bullet f'(z)$.

(4) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function and let $g : \mathbb{R}^3 \to \mathbb{R}$ be defined as $g(x, y, z) = f(x^2 + y^2 + z^2)$. Prove that,

$$||\nabla g(x, y, z)||^2 = 4(x^2 + y^2 + z^2) f'(x^2 + y^2 + z^2)^2$$

(5) Let $f : \Omega_n \to \mathbb{R}$ be a function with the property that if $\mathbf{x}, \alpha\mathbf{x} \in \Omega_n$ for some $\alpha \in \mathbb{R}$, then $f(\alpha\mathbf{x}) = \alpha^p f(\mathbf{x})$, for a fixed $p \in \mathbb{R}$. If f is differentiable at $\mathbf{x} \in \Omega_n$, show that $\mathbf{x} \bullet \nabla f(\mathbf{x}) = p f(\mathbf{x})$.