Homework 2, Math 4121, due 30 Jan 2014

(1) If $\{a_n\}$ is a decreasing sequence and $\sum a_n$ converges, prove that $\lim_{n\to\infty} na_n = 0$. This gives another proof that $\sum \frac{1}{n}$ diverges.

(2) Prove that $\sum_{p,\text{prime}} \frac{1}{p}$ diverges. (Hint: You may use $e^{2x} \geq (1-x)^{-1}$ if $x \in [0, \frac{1}{2}]$.)

(3) Show that the Cauchy product of $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n+1}}$ with itself is a divergent series.

(4) Prove by induction, $\prod_{k=2}^{n} \frac{k^3-1}{k^3+1} = \frac{2}{3}\frac{n^2+n+1}{n(n+1)}$. Deduce that $\prod_{k=2}^{\infty} \frac{k^3-1}{k^3+1} = \frac{2}{3}$.

(5) Prove that for any real number x,

$$\sin x = x \prod_{n=1}^{\infty} \cos \frac{x}{2^n}.$$

(6) Let $u_1 = \sqrt{2}$ and for $n \geq 2$, $u_n = \sqrt{2 + u_{n-1}}$ (where square roots always mean positive square roots). Prove that,

$$\frac{2}{\pi} = \prod_{n=1}^{\infty} \frac{u_n}{2}.$$

(Hint: Use the previous problem.) This is generally believed to be the first infinite product written down, in the 1500's.