Homework 3, Math 4121, due 6 Feb 2014

(1) Prove that if $f_n \to f, g_n \to g$ uniformly on a set S, so does $f_n + g_n \to f + g$. Give an example where $f_n g_n$ does not converge uniformly to fg.

(2) Let $\{f_n\}$ be a sequence of continuous functions on a compact interval monotonically decreasing (or increasing), converging pointwise to a continuous function f. Prove that the convergence is uniform.

Deduce using the above, if $\{f_n\}$ is a sequence of positive continuous functions on a compact interval and $\sum_{n=1}^{\infty} f_n$ converges pointwise to a continuous function, then the convergence is uniform.

(3) Let $X \subset \mathbb{R}^n$ be a closed subset and U an open subset containing X. Given a continuous function on X, show that it has an extension to all of $\mathbb{R}^n$, which is continuous and zero outside U.

(4) For any $k \in \mathbb{N} \cup \{0\}$ show that the series $\sum_{n=1}^{\infty} n^k x^n$ has a radius of convergence 1. Denoting this function by $f_k(x)$ in $(-1, 1)$, calculate $f_2(\frac{1}{2})$.

(5) Let $J(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{4^n (n!)^2}$ (Bessel function of order zero). Find its radius of convergence, and show that it is a solution of the differential equation $xy'' + y' + xy = 0$.

(6) (a) Assume $\sum_{n=0}^{\infty} a_n x^n$ has a radius of convergence $R > 0$. Further assume that $\sum_{n=0}^{\infty} a_n R^n$ also converges. Prove that $\sum_{n=0}^{\infty} a_n x^n$ converges uniformly on $[0, R]$.

(b) Let $\lim_{n\to\infty} a_n = L$ and $f(x)$ be the series $\sum a_n x^n$. Show that $\lim_{x \uparrow 1} (1 - x) f(x) = L$.