Homework 4, Math 4121, due 13 Feb 2014

(1) Prove that $\sum_{n=0}^{\infty}(n+1)z^n$ has radius of convergence 1 and thus defines a function $F(z)$ on $B(0,1)$ with $F(0)=1$. Find a power series expansion for $1/F(z)$ near the origin.

(2) Prove that $\sum_{n=1}^{\infty} \frac{x}{1+n^2x^2}$ converges pointwise for any $x>0$ and the convergence is uniform in $[r,\infty)$ for any $r>0$. So, we get a continuous function $F(x)$ for $x>0$ defined by this series. Calculate $\lim_{x\downarrow 0} F(x)$.

(3) Let f be a continuous function on $\mathbb{R}$ and assume that $\lim_{x\to+\infty} f(x) = A$, a finite number. What can you say about $\lim_{n\to\infty}\int_0^2 f(nx)dx$?

(4) Prove the formula $3\sum_{n=0}^{\infty}\frac{1}{(3n)!} = e + 2e^{-\frac{1}{2}}\cos\frac{\sqrt{3}}{2}$.

(5) Assume $f(z)=\sum_{n=0}^{\infty} a_n z^n$ is a powerseries (over complex numbers) with radius of convergence $R>0$. Prove that the closed set $\{a\in B(0,R) | f(a)=0\}$ is discrete (that is, has no accumulation points) unless $f(z)=0$ for all $z\in B(0,R)$.

(6) Consider the function $f(x)=\frac{1}{1+x^2}$ on $\mathbb{R}$. Prove that for any point $a\in\mathbb{R}$, f has a power series expansion $f(x)=\sum_{n=0}^{\infty} a_n(x-a)^n$ with radius of convergence $\sqrt{a^2+1}$.