Homework 5, Math 4121, due 20 Feb 2014

(1) (a) Let f be continuous on $[a, b]$, a compact interval. Show that there exists a polynomial P of degree at most one such that $(f + P)(a) = (f + P)(b) = 0$. Conclude that f can be extended to a continuous function on all of $\mathbb{R}$. (This avoids Tietze extension theorem, though in this case it is trivial).

(b) Further, if f is in $C^1([a, b])$ (this means, f is continuously differentiable in (a, b), the appropriate one sided limits exist for derivatives at a, b and the resulting function is continuous on $[a, b]$), show that there exists a polynomial P of degree at most three such that $f + P$ and its first derivatives vanish at a, b. Conclude that f can be extended to a function in $C^1(\mathbb{R})$. (Same argument can be used for C^k functions for any $k \in \mathbb{N}$ using polynomials of degree $2k+1$).

(2) (a) If f, g are continuous on $\mathbb{R}$ and one of them vanishes outside $[a, b]$, prove that $f * g$ is continuous.

(b) Let $f \in C^1(\mathbb{R})$ and vanish outside $[a, b]$ and let g be continuous on $\mathbb{R}$. Prove that $f * g \in C^1(\mathbb{R})$ and $(f * g)' = f' * g$. (Similar argument will also work for C^k functions).

(3) (a) Given $N \in \mathbb{N}$, a closed bounded interval $[a, b]$ and any k with $0 \leq k \leq N$, show that there is a polynomial P_k of degree at most N such that $\int_a^b P_k(x)x^j dx = 0$ for all $j \neq k$ with $0 \leq j \leq N$ and $\int_a^b P_k(x)x^k dx = 1$. (Rudimentary linear algebra could help.)

(b) Let polynomials P_n converge uniformly to f on $[a, b]$ and assume that $\deg P_n \leq N$ for a fixed N. Prove that f is a polynomial of degree at most N.

(4) Calculate (and describe) $f * f$ where f is the characteristic function of $[0, 1]$. That is, $f(x) = 1$ if $x \in [0, 1]$ and zero elsewhere.

(5) Let $A_m = \int_{-1}^1 (1 - x^2)^m dx$ and show that $A_m = A_{m-1} - \frac{A_m}{2m}$ for $m \geq 1$. (Integration by parts).