Homework 6, Math 4121, due 27, Feb 2014

equicontinuous=uniformly equicontinuous, for brevity in the following.

(1) If $\{f_n\}$ is an equicontinuous sequence of functions on a compact interval and $f_n \to f$ pointwise, prove that the convergence is uniform.

(2) If $|f_n(x) - f_n(y)| \leq M|x-y|^\alpha$ for some fixed M and $\alpha > 0$ and all x, y in a compact interval for all n, show that $\{f_n\}$ is equicontinuous.

(3) Let $\{f_n\}$ be a sequence of C^∞ functions on a compact interval such that for any integer $k \geq 0$, there exists an M_k such that $|f_n^{(k)}(x)| \leq M_k$ for all x and n. Prove that there exists a subsequence converging uniformly, together with all its derivatives to a C^∞ function.

(4) Prove that the set of all polynomials of degree at most N (fixed) and coefficients in $[-1, 1]$ is uniformly bounded and equicontinuous in any compact interval.

(5) Prove that the family of polynomials P of degree at most N with $|P(x)| \leq 1$ on $[0, 1]$ is equicontinuous on $[0, 1]$.

(6) Let $P_0 = 0$ and $P_{n+1}(x) = P_n(x) + \frac{x^2 - P_n^2(x)}{2}$.

   (a) Prove that

   $$|x| - P_{n+1}(x) = (|x| - P_n(x))\left(1 - \frac{|x| + P_n(x)}{2}\right).$$

   Deduce that $0 \leq P_n(x) \leq P_{n+1}(x) \leq |x|$ if $|x| \leq 1$.

   (b) Show that $|x| - P_n(x) \leq |x|(1 - \frac{|x|}{2})^n < \frac{2}{n+1}$ for $|x| \leq 1$.

   (c) Prove that $\{P_n\}$ converges uniformly to the function $g(x) = |x|$ in $[-1, 1]$.