Homework 7, Math 4121, due 6, March 2014

In the following, $(X, \mathcal{F})$ will be a measurable space.

(1) Let $f : X \to \mathbb{R}$ (or the extended real line) be a function. Prove that it is measurable if and only if for any rational number q, $\{x \in X | f(x) \geq q\} \in \mathcal{F}$.

(2) Let $Y \subset X$ and let $\mathcal{G} = \{A \in \mathcal{F} | A \subset Y\}$. Prove that $\mathcal{G}$ is a σ-algebra for Y.

(3) Prove that if f, g are measurable functions from $X \to \mathbb{R}$ and $c \in \mathbb{R}$, then cf is measurable and $f + g$ is measurable. Deduce that the set $\{x | f(x) \geq g(x)\}$ is measurable. (Note: If functions have range the extended real line, $f + g$ may not make sense at points for example where $f(x) = \infty, g(x) = -\infty$.)

(4) Prove that if f is measurable from $X \to \mathbb{R}$, so is f^2. Deduce that if f, g measurable, so is fg.

(5) If X is a finite set and $\mathcal{F}$ is the set of all subsets of X, describe all possible positive measures $\mu : \mathcal{F} \to [0, \infty]$.

(6) Let X be an uncountable set and let $\mathcal{F}$ be the collection of subsets $A \subset X$ such that either A or A^c is countable. Prove that $\mathcal{F}$ is a σ-algebra. Define $\mu : F \to [0, \infty]$ by $\mu(A) = 0$ if it is countable and $\mu(A) = 1$ if it is uncountable. Prove that μ is a positive measure.