Homework 8, Math 4121, due 20, March 2014

In the following, $(X, \mathcal{F}, \mu)$ will be a measure space.

(1) Let $f_1 \geq f_2 \geq \cdots \geq f \geq 0$ be a sequence of measurable functions with range $[0, \infty]$ and $\lim f_n = f$. Assume that $f_1 \in L^1(\mu)$. Then prove that $\lim \int_X f_n d\mu = \int_X f d\mu$. Give an example to show that the condition $f_1 \in L^1(\mu)$ is necessary.

(2) Let A be a measurable set and consider the sequence of functions, $f_n = \chi_A$ if n is odd and $f_n = 1 - \chi_A$ is n is even. Use this to deduce that strict inequality can occur in Fatou's lemma.

(3) Suppose $\mu(X) < \infty$ and $f_n : X \to \mathbb{R}$ is a sequence of bounded (that is, $|f_n| < M_n$ for constants M_n) measurable functions, converging uniformly to f. Prove that $\lim \int_X f_n d\mu = \int_X f d\mu$.

(4) Let $f : X \to [0, \infty]$ be measurable and let $A_1, A_2, \ldots$ be a countable collection of pairwise disjoint measurable sets and let $A = \cup A_n$. Prove that $\int_A f d\mu = \sum_{n=1}^{\infty} \int_{A_n} f d\mu$.

(5) Let $f \in L^1(\mu)$. Prove that, given $\epsilon > 0$, there exists a $\delta > 0$ such that if $A \in \mathcal{F}$ with $\mu(A) < \delta$ then $\int_A |f| d\mu < \epsilon$.

(6) Let $f_n : X \to \mathbb{R}$ be a sequence of measurable functions. Prove that the set of points where $\{f_n\}$ converge is a measurable set.