Homework 9, Math 4121, due 3, April 2014

In the following, $(X, \mathcal{F}, \mu)$ will be a measure space. As usual problem(s) in blue are not to be submitted.

(1) Let μ denote the Lebesgue measure on $[0,1]$. Given any r, $0 < r < 1$, show that there is a dense open set E of $[0,1]$ such that $\mu(E) = r$.

(2) Let $X = \mathbb{R}^n$ with the Lebesgue measure μ and let $f \in L^1(\mu)$ (which is usually written as $L^1(\mathbb{R}^n)$). Show that the set $\{x \in X | f(x) \neq 0\}$ is a countable union of measurable sets, each of which has finite measure. (One usually says that this set has σ-finite measure.)

(3) Calculate $\lim_{n\to\infty} \int_0^n (1+\frac{x}{n})^n e^{-2x} dx$. (Hint: Use the fact $\lim_{n\to\infty}(1+\frac{x}{n})^n = e^x$ and recall a proof if you have forgotten it.)

(4) Let $0 \leq f \leq 1$ be measurable on X. Prove that there exists measurable sets A_n such that $f = \sum_{n=1}^{\infty} \frac{\chi_{A_n}}{2^n}$.

(5) If $A \subset \mathbb{R}^n$ is Lebesgue measurable and given $\epsilon > 0$, prove that there exists $F \subset A \subset U$, with F, closed and U open with $\mu(U - F) < \epsilon$.

(6) Use the above two problems to prove the following. Let $X \subset \mathbb{R}^n$ be compact and f a Lebesgue measurable function on X with $0 \leq f \leq 1$. Given $\epsilon > 0$, there exists a continuous function g on X such that the set $\{x \in X | f(x) \neq g(x)\}$ has measure at most ϵ.

(7) Here is a problem to mull over. Let $\{f_n\}$ be a sequence of continuous functions on $X = [0,1]$ with $0 \leq f_n \leq 1$. Assume $\lim f_n(x) = 0$ for all x. Then, Lebesgue dominated convergence theorem tells us that $\lim \int_X f_n d\mu = 0$. Try proving it without using any measure theory.