(1) Decide whether the limits exist and find them when you can.

(a) \[ \lim_{n \to \infty} \frac{3n^2 + 4n + 1}{\sqrt{9n^4 + n^3 + n + 1}} \]

This limit is 1. Dividing the numerator and denominator by \(3n^2\), we get,

\[ \frac{3n^2 + 4n + 1}{\sqrt{9n^4 + n^3 + n + 1}} = \frac{1 + \frac{4}{3n} + \frac{1}{3n^3}}{\sqrt{1 + \frac{1}{9n} + \frac{1}{9n^2} + \frac{1}{9n^3}}} \]

Since all terms in the numerator and denominator other than the 1 go to zero as \(n \to \infty\), we see that the limit is 1.

(b) \[ \lim_{n \to \infty} \frac{5n}{n^2} \]

The limit goes to infinity as can be seen by applying L'Hopital's rule several times to the fraction \(\frac{5}{n^2}\). So, the limit does not exist.

(c) \[ \lim_{x \to 0} \frac{\sin 2x}{x^2} \]

Applying L'Hopital rule once, one can see that the limit goes to infinity and thus the limit does not exist.

(2) Decide which of the following series are convergent.

(a) \[ \sum_{n=0}^{\infty} \frac{n}{n^3 + 1} \]

Notice that the term for \(n = 0\) is zero and so we may as well look at the series \(\sum_{n=1}^{\infty} \frac{n}{n^3 + 1}\). There are many ways of doing it. The series converges. First, notice that all terms in the series are positive and for all \(n \geq 1\),

\[ \frac{n}{n^3 + 1} \leq \frac{n}{n^3} = \frac{1}{n^2}. \]

Thus, by comparison test (which can be applied) it is sufficient to show that \(\sum_{n=1}^{\infty} \frac{1}{n^2}\) is convergent. This is shown in the template using integral test.

(b) \[ \sum_{n=0}^{\infty} \frac{e^n}{n!} \]

This is just \(e^\pi\) and so the series converges.

(c) \[ \sum_{n=0}^{\infty} (-1)^n \frac{n^2}{n^3 + 1} \]

This is an alternating series and again we may drop the \(n = 0\) term since it is zero. We will show that the series converges by the alternating series test. If we let \(a_n = \frac{n^2}{n^3 + 1}\), then I claim that \(a_n \geq a_{n+1}\) for all \(n \geq 1\). (We could not have this for \(a_0\), since it is zero). This is obvious, but let me just write out the algebra. (Even if you do not, it is fine, as long as you know how to do it). For any \(n \geq 1\) we have,

\[ 1 > 1 - \frac{1}{(n+1)^2} \geq \frac{1}{n^2} - \frac{1}{(n+1)^2}. \]
Adding \( n \) to both sides and transferring a term from right to left, we get,

\[
 n + 1 + \frac{1}{(n + 1)^2} > n + \frac{1}{n^2}
\]

and thus

\[
\frac{1}{n + 1 + \frac{1}{(n + 1)^2}} < \frac{1}{n + \frac{1}{n^2}}.
\]

Multiplying the numerator and denominator of the first fraction by \((n + 1)^2\), we get \( a_{n+1} \) and similarly for the second. This shows \( a_n > a_{n+1} \).

Also, one sees that

\[
\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{1}{n + \frac{1}{n}} = 0.
\]

So, by the alternating series test, the series converges. (d) \( \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \)

The series converges. We can write

\[
a_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}
\]

and thus if we look at the partial sum,

\[
S_k = \sum_{n=1}^{k} \frac{1}{n(n+1)}
= \sum_{n=1}^{k} \left( \frac{1}{n} - \frac{1}{n+1} \right)
= (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + \cdots + (\frac{1}{k} - \frac{1}{k+1})
= 1 - \frac{1}{k+1}
\]

Recall that a series converges just means that sequence of partial sums converge. So, we only need to see whether the sequence \( S_k \) converges. Clearly,

\[
\lim_{k \to \infty} 1 - \frac{1}{k+1} = 1
\]

and so the series converges (to 1).

(3) Decide the interval of convergence for the following series.
(a) \( \sum_{n=1}^{\infty} \frac{x^n}{(n!)^3} \)

The series converges for all values of \( x \). We will use ratio test. So, let

\[ \rho_n = \left| \frac{x^{n+1}}{(n+1)^3} \right| = \left| \frac{x}{(n+1)^3} \right| \]

Then for any value of \( x \), we see that \( \lim_{n \to \infty} \rho_n = 0 \) and so the series converges for all values of \( x \), by ratio test.

(b) \( \sum_{n=1}^{\infty} \frac{nx^n}{n+1} \)

Again, let us use the ratio test to see what we get.

\[ \rho_n = \left| \frac{(n+1)x^{n+1}}{n+2} \right| = \left| \frac{(n+1)^2x}{n(n+2)} \right| \]

So, \( \lim_{n \to \infty} \rho_n = |x| \) and thus we get that if \( |x| < 1 \) the series converges and if \( |x| > 1 \), the series diverges. Thus we are left to check the convergence at the two end points \( x = \pm 1 \). If \( x = 1 \), the series is \( \sum \frac{n}{n+1} \) and since this is a series with positive terms and \( \lim \frac{n}{n+1} = 1 \neq 0 \), by the preliminary test, we know that the series must diverge. Similarly if \( x = -1 \), our series is \( \sum (-1)^n \frac{n}{n+1} \) and since \( \lim(-1)^n \frac{n}{n+1} = 1 \neq 0 \), by preliminary test, we see that the series diverges.

So, the interval of convergence is the open interval \((-1, 1)\).

(c) \( \sum_{n=1}^{\infty} (-1)^n n^2 x^n \)

Again, let us use the ratio test.

\[ \rho_n = \left| \frac{(-1)^{n+1}(n+1)^2x^{n+1}}{(-1)^nn^2x^n} \right| = \left| \frac{(n+1)^2x}{n^2} \right| \]

Thus, \( \lim \rho_n = |x| \) and so we see as before that the series converges if \( |x| < 1 \) and diverges for \( |x| > 1 \). Again, we need to check what happens when \( x = \pm 1 \). If \( x = 1 \), our series is just \( \sum n^2 \) and by preliminary test, it does not converge. Similarly, if \( x = -1 \), our series is \( \sum (-1)^n n^2 \) and by preliminary test, we see that the series does not converge.

So, the interval of convergence is \((-1, 1)\).

(4) Find the first few terms (at least four terms) of the Maclaurin series for the following functions.

(a) \( f(x) = \int_0^x e^{-t^2} \, dt \)
The terms are just $f^{(k)}(0)$. Clearly $f(0) = 0$ and $f'(x) = e^{-x^2}$ and thus $f'(0) = 1$. I will leave you to check $f^{(2)}(x)$ and $f^{(3)}(x)$.

(b) $\frac{x}{\sin x}$

If we let $g(x) = \frac{x}{\sin x}$, then $g(0) = 1$ (by L’Hopital’s rule).

$g'(x) = \frac{\sin x - x \cos x}{(\sin x)^2}$. One can calculate $g'(0)$ using L’Hopital or using Maclaurin series.

Let us first do this using L’Hopital (and I will leave you to check that I can appeal to it).

$$g'(0) = \lim_{x \to 0} g'(x)$$

$$= \lim_{x \to 0} \frac{\cos x - \cos x + x \sin x}{2 \sin x \cos x}$$

$$= \lim_{x \to 0} \frac{x}{2 \cos x} = 0$$

If we used series,

$$\sin x - x \cos x = (x - \frac{x^3}{3!} + \cdots) - x(1 - \frac{x^2}{2!} + \cdots)$$

$$= x^3 h(x)$$

where $h(x)$ is a power series.

On the other hand, $\sin x = x k(x)$ where $k(x)$ is a power series with $k(0) = 1$. Thus,

$$g'(x) = \frac{\sin x - x \cos x}{(\sin x)^2} = \frac{x h(x)}{k(x)}$$

and thus $g'(0) = 0$. I will leave you to check the next two coefficients.