Solutions to Homework 11, Math 308

(1) Write down series solutions for the following differential equations.

(a) \( y' - y = f(x) \) where \( f(x) = \sum_{n=0}^{\infty} a_n x^n \) with initial condition \( y(0) = 0 \).

Let \( y = \sum b_n x^n \) be a series solution. The initial condition says that \( b_0 = 0 \). Substituting in the differential equation, we get,

\[
\sum_{n=1}^{\infty} nb_n x^{n-1} - \sum b_n x^n = \sum a_n x^n.
\]

Equating coefficient of \( x^n \), we get, \( (n+1)b_{n+1} - b_n = a_n \). Solving this recursive equations, one gets \( b_n = \frac{n}{n(n-1) \cdots (n-i+1)}, n = 1, 2, \ldots \).

(b) \( x^2 y'' + xy' + y = 0 \).

Easy to see that the only series solution is \( y = 0 \).

(2) Calculate \( P_3(x) \) and \( P_4(x) \), the third and fourth Legendre polynomials.

\( P_3(x) = \frac{5x^3 - 3x}{2} \) and \( P_4(x) = \frac{35x^4 - 30x^2 + 3}{8} \).

(3) If \( F(x), A(x) \) are polynomials and \( 0 \leq k \leq n \) are integers, show that we can write \( \frac{d^k}{dx^k} F_n(x) A(x) \) as \( F_{n-k}(x) G(x) \) for a polynomial \( G(x) \).

This is trivial by induction on \( k \), the case \( k = 0 \) being obvious. If \( k > 0 \), we can write \( D^k(F_n A) \) (where \( D = \frac{d}{dx} \)), as

\[
D^{k-1}(DF^n A) = D^{k-1}(F_{n-1}(nF' A + Fa')).
\]

(4) Find the Legendre series for the function \( f(x) = 0, -1 < x < 0 \) and \( f(x) = x, 0 < x < 1 \).

If \( f(x) = \sum_{n=0}^{\infty} a_n P_n(x) \), \( a_0 = 1/4, a_1 = 1/2 \). If \( n \geq 2 \), we see that \( a_n = 0 \) if \( n \) is odd and if \( n \geq 2 \) is even,

\[
a_n = \frac{(-1)^{n-2} \binom{n-2}{n/2} (n/2)!}{2^{n+1} n!}.
\]

(5) Find the Legendre series for \( f(x) = P'_n(x) \).

Writing \( P'_n(x) = \sum a_m P_m(x) \), one sees immediately that \( a_m = 0 \) for \( m \geq n \). If \( m < n \), integrating by parts, one sees that \( a_m = 0 \) if \( m + n \) is even and if \( m + n \) is odd, we get \( a_m = 2m + 1 \).

(6) Let \( \Phi(x, h) = \sum P_n(x) h^n \) be the generating function for Legendre Polynomials. Show that \( (x - h) \frac{\partial}{\partial \Phi} h = h \frac{\partial}{\partial \Phi} \).

This is easy by using the fact that \( \Phi(x, h) = (1 - 2xh + h^2)^{-1/2} \).