Solutions to Homework 3, Math 308, Spring 2010

(1) For $u = e^x \cos y$ verify that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$ and $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

We calculate all the required derivatives.

\[
\frac{\partial u}{\partial x} = u, \\
\frac{\partial u}{\partial y} = -e^x \sin y \\
\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x} \right) = u \\
\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial y} \right) = -e^x \sin y \\
\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial x} \right) = -e^x \sin y \\
\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial y} \right) = -e^x \cos y = -u
\]

The rest is clear.

(2) For a function $f(x, y)$ in two variables let the Maclaurin series be $f(x, y) = \sum_{i,j=0}^{\infty} a_{ij} x^i y^j$. Find expressions for $a_{ij}$ in terms of $i, j$ for the following functions.

(a) $f(x, y) = \sin(x + y)$

It is convenient to introduce the notation $[a/2]$ for integers $a$ to mean the largest integer less than or equal to $a/2$. For example,

\[
\left[ \frac{0}{2} \right] = 0, \quad \left[ \frac{1}{2} \right] = 0, \quad \left[ \frac{2}{2} \right] = 1, \quad \left[ \frac{7}{2} \right] = 3, \quad \text{etc.}
\]

Then $a_{ij} = 0$ if $i + j$ is even. If $i + j$ is odd, then,

\[a_{ij} = (-1)^{[i+j]/2} \frac{1}{i! j!}.
\]

(b) $f(x, y) = \frac{1}{x+y}$

$a_{ij} = 0$ if $i \neq j$ and $a_{ii} = 1$ for all $i$.

(3) The thin lens formula is $i^{-1} + o^{-1} = f^{-1}$, where $f$ is the focal length, $o$ and $i$, the distances from the lens to the object and image respectively. If $i = 15$ when $o = 10$, use differentials to find $i$ when $o = 10.1$, for a given lens.

Taking differentials, we get $i^{-2} di + o^{-2} do = -d(f^{-1}) = 0$ since the focal length is constant, and thus, we have, by putting,
\[ i = 15, o = 10 \text{ and } do = 10.1 - 10 = 0.1, \text{ we get } di = -0.225 \]
and thus the value for \( i \) is \( 15 - 0.225 = 14.775 \).

(4) Use differentials to estimate the change in
\[ f(x) = \int_0^x \frac{e^{-t}}{t^2 + 0.51} dt \]
if \( x \) changes from 0.7 to 0.71.
Here, \( df(x) = \frac{e^{-x}}{x^2 + 0.51} dx \) and putting \( x = 0.7 \) and \( dx = 0.01 \),
we can calculate \( df(x) \), which is just \( e^{-0.7}/100 \).

(5) Given that \( z = (x + y)^5 \), and \( y = \sin 10x \), find \( \frac{dz}{dx} \).
We have, by chain rule, if we put \( u = x + y \),
\[ \frac{dz}{dx} = \frac{dz}{du} \cdot \frac{du}{dx}. \]
\[ \frac{dz}{du} = 5u^4 = 5(x + y)^4 \text{ and } \frac{du}{dx} = 1 + 10 \cos 10x. \text{ Thus, } \frac{dz}{dx} = 5(x + y)^4(1 + 10 \cos 10x). \]

(6) If \( P_i \)'s are finitely many points in the plane with masses \( m_i \) at
these points, the formula for the moment of inertia at a point \( P \)
is given by \( \sum m_i d(P_i, P)^2 \), where \( d(P_i, P) \) is the distance
from \( P_i \) to \( P \). Given masses \( m_i \) at \( P_i = (x_i, y_i) \) find the co-ordinates
of the point \( P \) where the moment of inertia is least.
If the co-ordinates of \( P \) are \( (x, y) \), then we want to minimize
the function, \( M(x, y) = \sum m_i (x - x_i)^2 + \sum m_i (y - y_i)^2 \). So, we
equate the two partial derivatives to zero to get,
\[ \sum m_i (x - x_i) = 0 \quad \sum m_i (y - y_i) = 0 \]
yielding
\[ x = \frac{\sum m_i x_i}{\sum m_i}, \quad y = \frac{\sum m_i y_i}{\sum m_i}. \]
It is clear that this point should give the minimum (and not
a maximum). If you are not convinced, you may use the second
derivative test. But, from the physical situation, clearly this can
not be a maximum, since if you choose \( x, y \) large, the moment
of inertia clearly increases without limit.

(7) If we made a can with a circular base and top , whose total
(outer) surface area is \( 24\pi \) square meters, what would be the
maximum possible volume?
If \( r \) is the radius of the can and \( h \) its height, then the total
surface area is \( A(r, h) = 2\pi rh + 2\pi r^2 \) and its volume is \( V(r, h) = \pi r^2 h \). If we use the Lagrange multiplier technique, we look at
\( F(r, h) = V - \lambda A \) and put the derivatives of \( F \) with respect to
\( r, h \) zero and using the fact \( A = 24\pi \) one can easily solve these
to get, $\lambda = 1, r = 2, h = 4$ and thus, $V = 16\pi$ cubic meters. Again, for physical reasons or otherwise, one can easily see that this is the maximum (and not a minimum).