Solutions to Homework 4, Math 308, Spring 2010

(1) Given \( x^2 + y^2 = 2st - 10 \) and \( 2xy = s^2 - t^2 \), find \( \frac{\partial x}{\partial s} \), \( \frac{\partial x}{\partial t} \), \( \frac{\partial y}{\partial s} \) and \( \frac{\partial y}{\partial t} \) at \((x, y, s, t) = (4, 2, 5, 3)\).

We differentiate the two equations with respect to \( s, t \) to get four equations.

\[
\begin{align*}
2x \frac{\partial x}{\partial s} + 2y \frac{\partial y}{\partial s} &= 2t \\
2x \frac{\partial x}{\partial t} + 2y \frac{\partial y}{\partial t} &= 2s \\
2y \frac{\partial x}{\partial s} + 2x \frac{\partial y}{\partial s} &= 2s \\
2y \frac{\partial x}{\partial t} + 2x \frac{\partial y}{\partial t} &= -2t
\end{align*}
\]

Substituting the values for \( x, y, s, t \), we get four linear equations in the various partial derivatives, which can easily be solved to get,

\[
\frac{\partial x}{\partial s} = \frac{1}{6}, \quad \frac{\partial y}{\partial s} = \frac{7}{6}, \quad \frac{\partial x}{\partial t} = \frac{13}{6}, \quad \frac{\partial y}{\partial t} = -\frac{11}{6}.
\]

(2) Given \( f(x, y, z) = 0 \) and \( g(x, y, z) = 0 \), find a formula for \( \frac{dy}{dx} \).

Differentiating, we get two linear equations,

\[
\begin{align*}
f_x dx + f_y dy + f_z dz &= 0 \\
g_x dx + g_y dy + g_z dz &= 0
\end{align*}
\]

Multiplying the first by \( g_z \) and the second by \( f_z \) and subtracting, we get,

\[
(f_x g_z - f_z g_x) dx + (f_y g_z - f_z g_y) dy = 0
\]

and thus,

\[
\frac{dy}{dx} = -\frac{f_x g_z - f_z g_x}{f_y g_z - f_z g_y}.
\]

(3) Solve for \( z \) as a function of \( x, y \), given,

\[2 \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 10 \frac{\partial^2 z}{\partial y^2} = 0.\]

As we did in class, we factorize,

\[2x^2 + xy - 10y^2 = (2x + 5y)(x - 2y).\]

Thus, the differential equation is,

\[\left(2 \frac{\partial}{\partial x} + 5 \frac{\partial}{\partial y}\right) \left(\frac{\partial}{\partial x} - 2 \frac{\partial}{\partial y}\right) z = 0.\]
Thus we change variables,

\[ x = 2u + v, \quad y = 5u - 2v \]

and then the equation becomes, \( \frac{\partial^2 z}{\partial u \partial v} = 0 \). Thus the solutions are \( z = P(u) + Q(v) \) for functions \( P, Q \) and so the solutions are, \( z = P(2x + y) + Q(5x - 2y) \) for some functions \( P, Q \) in one variable.

(4) Express \( z \) as a function of \( x, y \), given,

\[ \frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = 1. \]

As before, we see that the equation is just,

\[ \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \right) \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) z = 1. \]

We change variables, \( x = u + v, y = -u + v \) and then the differential equation is just, \( \frac{\partial^2 z}{\partial u \partial v} = 1 \), whose solutions are \( z = uv + P(u) + Q(v) \) for some functions \( P, Q \). Thus, the solutions are

\[ z = \frac{x^2 - y^2}{4} + P(x - y) + Q(x + y) \]

for some functions \( P, Q \) in one variable.

(5) Calculate \( dy/dx \) if \( y = \int_{x^2}^{3x^3} \sin t \, dt \).

\[ 3x^2 \sin x^3 - 2x \sin x^2. \]

(6) Calculate,

\[ \lim_{x \to 0} \frac{1}{x} \int_0^x \frac{1 - \cos t}{t^2} \, dt. \]

If we denote by \( I(x) = \int_0^x \frac{1 - \cos t}{t^2} \, dt \), notice that \( I(0) = 0 \). So, we may apply L’Hopital’s rule to get that the above quantity is just,

\[ \lim_{x \to 0} \frac{dI(x)/dx}{x} = \lim_{x \to 0} \frac{dI(x)}{dx}. \]

\[ \frac{dI(x)}{dx} = \frac{1 - \cos x}{x^2} \]

By applying L’Hopital or using series, one immediately sees that the answer is \( 1/2 \).

(7) Given \( \int_0^\infty e^{-at^2} \, dt = \frac{1}{2} \sqrt{\pi/a} \), calculate \( \int_0^\infty t^2 e^{-at^2} \, dt \).

We differentiate the integral with respect to \( a \) to get,

\[ \int_0^\infty t^2 e^{-at^2} \, dt = \frac{1}{4} \sqrt{\pi/a^3}. \]
(8) Show that \( \frac{d}{dx} \int_{\cos x}^{\sin x} \sqrt{1-t^2} \, dt = 1. \)

This derivative is just,

\[
\cos x \sqrt{1 - \sin^2 x} + \sin x \sqrt{1 - \cos^2 x} = \cos^2 x + \sin^2 x = 1.
\]