Solutions to Homework 5, Math 308, Spring 2010

(1) Calculate \( \int \int (9 + 2y^2)^{-1} \, dx \, dy \) over the quadrilateral with vertices \((1, 3), (3, 3), (2, 6), (6, 6)\).

\[ \frac{\log 3}{6} \]

(2) Find the volume in the first octant bounded by the coordinate planes and the plane \( x + 2y + z = 4 \).

\[ \frac{64}{3} \]

(3) Calculate the following integral by changing the order of integration (since it cannot be done as it stands, easily).

\[ \int_{x=0}^{2} \int_{y=x}^{2} e^{-y^2/2} \, dy \, dx \]

\[ 1 - e^{-2} \]

(4) Let \( S \) be a circle with center \((0, 4)\) and radius \( r < 4 \) in the \( xy\)-plane. Calculate the volume enclosed and the surface area when it is revolved around the \( x\)-axis. (This object is called a torus, which is doughnut shaped).

Volume = \( 8\pi^2 r^2 \) and the surface area = \( 16\pi r \).

(5) Let \( y = f(x), a \leq x \leq b \) be an arc where we assume that \( f(x) > 0 \) for all \( x \) in this interval. Write down integrals expressing the area under the arc (that is between the arc and the \( x\)-axis), the arc length, the volume of the solid generated when revolved about the \( x\)-axis and the surface area of this solid of revolution.

The area is given by \( \int_{a}^{b} f(x) \, dx \). The arc length is given by \( \int_{a}^{b} \sqrt{1 + (df/dx)^2} \, dx \). The volume of the solid of revolution is \( \int_{a}^{b} \pi f(x)^2 \, dx \). The surface area is \( \int_{a}^{b} 2\pi f(x) \sqrt{1 + (df/dx)^2} \, dx \).

(6) Find the volume inside the cone \( 3z^2 = x^2 + y^2 \), above the plane \( z = 2 \) and inside the sphere of radius six with center the origin, using spherical coordinates.

Here I will just describe the region and let you evaluate the integral (which you should be able to do knowing \( \int \sec^3 \theta \, d\theta \)).

The interior of the cone can be described in spherical coordinates by \( \theta \leq \pi/3 \). The sphere of course is just \( r \leq 6 \). That we want \( z = r \cos \theta \geq 2 \) means, the region can be described by, \( 2/\cos \theta \leq r \leq 6, 0 \leq \theta \leq \pi/3 \) and \( 0 \leq \phi \leq 2\pi \). The integral
therefore you want to evaluate is,

\[ \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi/3} \int_{r=a/cos\theta}^{2} r^2 \sin \theta drd\theta d\phi. \]

(7) Compute the gravitational attraction on a unit mass at the origin due to the mass (of constant density, say \( \rho \)) of a body which is bounded by the sphere \( r = 2a \) and above the \( z = a \) plane. (Denote by \( G \), the gravitational constant, so that the force between two point masses \( m_1, m_2 \) situated at a distance \( r \) is \( Gm_1m_2/r^2 \)).

If \( \mathbf{F} \) is the force vector, by symmetry, it is clear that its components in the \( x, y \) directions are zero. So, \( \mathbf{F} = F\mathbf{k} \), where \( F \) is its magnitude. So, we only need to calculate the magnitude of the force. If we have a mass \( dM \) situated at the point \( (r, \theta, \phi) \), then the force at the origin is just \( (G/r^2)dM \).

The region of interest can be described as \( 0 \leq \phi \leq 2\pi \), \( 0 \leq \theta \leq \pi/3 \) and \( a/cos\theta \leq r \leq 2a \). Since \( dM \) is volume times the density, we see that \( dM = \rho r^2 \sin \theta drd\theta d\phi \). So, the integral you want to evaluate is,

\[ \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi/3} \int_{r=a/cos\theta}^{2a} G\rho \sin \theta drd\theta d\phi. \]