Answers to Homework 7, Math 308

1. Compute the divergence and curl of the vector field \( \mathbf{V} = x \sin y \mathbf{i} + \cos y \mathbf{j} + xy \mathbf{k} \).

   \[
   \text{div} \, \mathbf{V} = 0 \quad \text{and} \quad \text{curl} \, \mathbf{V} = xi - yj - x \cos yk.
   \]

2. Calculate the Laplacian \( \nabla^2 \left( \frac{1}{\sqrt{x^2 + y^2 + z^2}} \right) \).

   This is zero.

3. Calculate the line integral \( \int \frac{xdy - ydx}{x^2 + y^2} \) along the following path. Start from (1, 0), go along the x-axis to (a, 0) where \( a > 0 \), then go counterclockwise along the semicircle with radius \( a \), ending at \((-a, 0)\) and go along the x-axis to \((-1, 0)\).

4. If \( C \) is any closed loop in the plane, show that \( \oint_C y \cos xy \, dx + x \cos xy \, dy = 0 \).

   This is just an application of Green’s theorem. If \( R \) is the region enclosed by \( C \) (rigorously speaking, you may have to look at several pieces if the curve crosses itself), then we have

   \[
   \oint_C y \cos xy \, dx + x \cos xy \, dy = \iint_R \left( \frac{\partial(y \cos xy)}{\partial x} - \frac{\partial(x \cos xy)}{\partial y} \right) \, dx \, dy
   = \iint_R (\cos xy - xy \sin xy - \cos xy + xy \sin xy) \, dx \, dy
   = \iint_R 0 \, dx \, dy = 0
   \]

5. For the force field \( \mathbf{F} = (y + z) \mathbf{i} - (x + z) \mathbf{j} + (x + y) \mathbf{k} \), find the work done in moving a particle around the circle \( x^2 + y^2 = 1, z = 0 \) moving counterclockwise.

   \( -2\pi \).

6. Show that the electric field \( \mathbf{E} = q \frac{\mathbf{r}}{r^3} \) is conservative and find a scalar potential \( \phi \) with \( \mathbf{E} = -\nabla \phi \).

   We can take \( \phi = \frac{q}{r} \) in spherical co-ordinates. (Of course this is well defined only upto adding a constant).

7. Calculate \( \oint 2y \, dx - 3x \, dy \) around the square with vertices \((3, 1), (5, 1), (5, 3)\) and \((3, 3)\), without integration.

   By Green’s theorem, if we denote the region by \( R \) and the closed curve consisting of the edges of the square \( C \), then,

   \[
   \oint 2y \, dx - 3x \, dy = \iint_R \left( \frac{\partial(-3x)}{\partial x} - \frac{\partial(2y)}{\partial y} \right) \, dx \, dy
   = \iint_R (-5) \, dx \, dy
   \]

   This is just \(-5\) times the area of the square which is 4 and thus the result is \(-20\).