Homework 6, Math 308, due March 22nd

(1) Let $\mathbf{A} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ be a vector with its tail at the origin. Assume that it rotates around the x-axis with angular velocity $\boldsymbol{\omega}$. Calculate (as a vector) the linear velocity of the head of $\mathbf{A}$.

(2) Prove the following for arbitrary vectors as specified.
   (a) $(\mathbf{A} \cdot \mathbf{B})^2 \leq |\mathbf{A}|^2 |\mathbf{B}|^2$.
   (b) $(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C})$. (This is called Lagrange's identity).
   (c) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) + \mathbf{B} \times (\mathbf{C} \times \mathbf{A}) + \mathbf{C} \times (\mathbf{A} \times \mathbf{B}) = 0$. (This is called Jacobi's identity).

(3) Let the position vector of a particle at time t be given by $\mathbf{r} = \mathbf{i} \cos t + \mathbf{j} \sin t + \mathbf{k}t$. Show that $|\mathbf{v}|$ and $|\mathbf{a}|$ are constant, where $\mathbf{v}$ is the velocity and $\mathbf{a}$ is the acceleration.

(4) The position vector of a particle is $\mathbf{r} = r(t)\mathbf{e}_r$ in polar coordinates. (This is not quite precise from a mathematical point of view, but in Physics, it seems common and they manipulate this formally). Find its velocity and acceleration in polar coordinates.

(5) Show that in polar coordinates, $\boldsymbol{\nabla} f = \mathbf{e}_r \frac{\partial f}{\partial r} + \mathbf{e}_\theta \frac{1}{r} \frac{\partial f}{\partial \theta}$.

(6) Let $f = e^x \cos y$ represent the temperature at a point (x, y). Find the direction in which the temperature is increasing most rapidly at $(1, -\pi/4)$ and the magnitude of the rate of increase.