Template for answers

Do your scratchwork elsewhere. They have no place in the answers. Answers are for communication and so follow rules of English and be precise and linear. Always apprise the reader why you do a calculation before you do it so that the reader can feel what to expect. Be convincing and state the facts that you are using for your deductions.

(1) Find $\lim_{x\to 0} \frac{\tan x}{x}$.

We will show that this limit is 1. Since

$$\frac{\tan x}{x} = \frac{\sin x}{x} \times \frac{1}{\cos x},$$

we get that,

$$\begin{aligned} \lim_{x\to 0} \frac{\tan x}{x} &= \lim_{x\to 0} \frac{\sin x}{x} \times \frac{1}{\cos x} \\ &= \lim_{x\to 0} \frac{\sin x}{x} \times \lim_{x\to 0} \frac{1}{\cos x} \\ &= 1 \times 1 = 1 \end{aligned}$$

Another way to do this is as follows.

We will use L'Hopital's rule. Since $\tan 0 = 0$ and the denominator is also zero at $x = 0$, we can use this rule. (*Always say why you can use a particular fact*). The derivative of $\tan x$ is $\sec^2 x$ and the derivative of x is 1. Thus, we get,

$$\begin{aligned} \lim_{x\to 0} \frac{\tan x}{x} &= \lim_{x\to 0} \frac{\sec^2 x}{1} \\ &= \frac{\sec^2 0}{1} = 1 \end{aligned}$$

(2) Decide when the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges where p is any real number.

We will show that the series diverges for $p \leq 1$ and converges for $p > 1$ by using the integral test.

So, we consider the function $f(x) = x^{-p}$ defined for $x \geq 1$. Then our series is just $\sum_{n=1}^{\infty} f(n)$ and $f(x)$ is a non-increasing function. So, we can use the integral test. (*Make sure when you can apply a test*).

We first look at the case $p < 1$. Then,

$$\begin{aligned} \int_1^{\infty} x^{-p} dx &= \frac{x^{-p+1}}{-p+1}\Big|_1^{\infty} \\ &= \infty \end{aligned}$$

since $-p + 1 > 0$. Thus by integral test, the series diverges for $p < 1$.

Next let us look at the case when $p = 1$. Then,

$$\int_1^\infty x^{-1} dx = \log x \Big|_1^\infty$$
$$= \infty$$

Again, by integral test, the series diverges.

Finally, let us look at the case $p > 1$. Then,

$$\int_1^\infty x^{-p} dx = \frac{x^{-p+1}}{-p+1}\Big|_1^\infty$$
$$= \frac{1}{p-1}$$

because, when $-p + 1 < 0$, we have $\lim_{x\to\infty} x^{-p+1} = 0$. So, by integral test, we see that the series converges for $p > 1$.