1.1,1.2

Tuesday, September 15, 2020 3:07 PM

<u>Sec 1.1</u> Exer: 12

$$x_1 - 3x_2 + 4x_3 = -4 \quad — (1)$$

$$3x_1 - 7x_2 + 7x_3 = -8 \quad — (2)$$

$$-4x_1 + 6x_2 - x_3 = 7 \quad — (3)$$

We try to get rid of x_1 from (2) & (3) by row operations.

Since coefficient of x_1 in (2) is 3, and 1 in (1), we multiply (1) by 3 and subtract from (2)

$$3x_1 - 7x_2 + 7x_3 = -8 \qquad (2)$$

$$3x_1 - 9x_2 + 12x_3 = -12 \qquad 3\times(1).$$

$$0\cdot x_1 + 2x_2 - 5x_3 = 4 \qquad (2) - 3\times(1)$$

Similarly,

$$-4x_1 + 6x_2 - x_3 = 7 \quad (3)$$

$$4x_1 - 12x_2 + 16x_3 = -16 \quad 4 \times (1)$$

$$0 \cdot x_1 - 6x_2 + 15x_3 = -9 \quad (3) + 4 \times (1).$$

Thus the 3 equations become.

$$x_1 - 3x_2 + 4x_3 = -4 \quad (1) = (1').$$

$$2x_2 - 5x_3 = 4 \quad (2')$$

$$-6x_2 + 15x_3 = -9 \quad (3').$$

This system is equivalent to our original system.

(2') and (3') involve no x_1.

So we use similar row operations using (2') & (3') and eliminate x_2 from (3').

$$-6x_2 + 15x_3 = -9 \quad (3')$$

$$6x_2 - 15x_3 = 12 \quad (2') \times 3$$

$$6x_2 - 15x_3 = 12 \quad (2')\times 3$$

$$0x_2 + 0\cdot x_3 = 3 \quad (3') + 3\times(2')$$

So we have an equivalent system:

$$x_1 - 3x_2 + 4x_3 = -4$$
$$2x_2 - 5x_3 = 4$$
$$0 = 3$$

The last equation is never true, so our system has no solutions. i.e. inconsistent.

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<u>Exer 14</u>:

$$x_1 - 3x_2 = 5 \quad (1)$$
$$-x_1 + x_2 + 5x_3 = 2 \quad (2)$$
$$x_2 + x_3 = 0 \quad (3).$$

Replace (2) by (2) + (1). (Do you see why?). We get an equivalent system:

$$\begin{aligned} x_1 - 3x_2 \quad\quad\quad &= 5 \\ -2x_2 + 5x_3 &= 7 \\ x_2 + x_3 &= 0. \end{aligned}$$

For arithmetic ease, we flip ② & ③ :

$$\begin{aligned} x_1 - 3x_2 \quad\quad\quad &= 5 \\ x_2 + x_3 &= 0 \\ -2x_2 + 5x_3 &= 7. \end{aligned}$$

$$\begin{aligned} x_1 - 3x_2 \quad\quad\quad &= 5 \\ x_2 + x_3 &= 0 \\ 7x_3 &= 7. \end{aligned}$$

Last equation gives

$$\boxed{x_3 = 1}$$

Substituting in ②, we get

Substituting in (2), we get

$$x_2 + 1 = 0 \text{ or } \boxed{x_2 = -1},$$

Substituting in (1) gives

$$x_1 + 3 = 5 \text{ or } \boxed{x_1 = 2}.$$

So our unique solution is,

$$x_1 = 2, \; x_2 = -1, \; x_3 = 1$$

(You must check this is indeed a solution to our original system of equations.)

Exer 27:

$$x_1 + 3x_2 = f \quad \text{— (1)}$$

$$cx_1 + dx_2 = g \quad \text{— (2)}$$

We do the row operation,

(2) $- c \times$ (1) to get:

$$x_1 + 3x_2 = f$$

$$(d-3c)x_2 = g-3f$$

Since f, g can be arbitrary, $g-3f$ can be non-zero.

Then the last equation has a solution only if $d-3c \neq 0$. (Remember this from the first class?)

So $\boxed{d-3c \neq 0}$ is a necessary condition for this system to be consistent for any choice of f and g.

I claim this is also sufficient. Then, we have from ~~the~~ last eqn:

$$\boxed{x_2 = \frac{g-3f}{d-3c}}$$

Substitute in 1 to get,

$$x_1 + 3 \cdot \frac{g-3f}{d-3c} = f$$

or $\boxed{x_1 = f - 3 \cdot \frac{g-3f}{d-3c}}$

This $d-3c \neq 0$ is necessary and sufficient for the system to be consistent for any choice of f and g.

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See 1.2 Exer 4:

$$\begin{bmatrix} 1 & 3 & 5 & 7 \\ 3 & 5 & 7 & 9 \\ 5 & 7 & 9 & 1 \end{bmatrix}$$

Subtract 3×1^{st} row from 2^{nd} row and 5×1^{st} row from 3^{rd} row:

(So the first column is the pivot column & 1 is the pivot.)

$$\rightsquigarrow \begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & -4 & -8 & -12 \\ 0 & -8 & -16 & -34 \end{bmatrix}$$

Now we "hide" the first row and then the second column is the pivot column (first non-zero column.) Then we can use -4 as the pivot. So ~~substract~~ subtract $2 \times$ second row from the 3^{rd} to get:

$$\begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & -4 & -8 & -12 \\ 0 & 0 & 0 & -10 \end{bmatrix}$$

Now hide 1^{st} & 2^{nd} row, and the last row has 4^{th} column as the pivot column.

This is the echelon form. To get to the reduced echelon form we do reverse row operations.

We multiply the last row

by $-\frac{1}{10}$ to get

$$\begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & -4 & -8 & -12 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Next we subtract 7×3^{rd} row from 1^{st} row, add 12×3^{rd} row to the 2^{nd} row to get:

$$\begin{bmatrix} 1 & 3 & 5 & 0 \\ 0 & -4 & -8 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Next, the pivot is -4, so multiply by $\frac{1}{-4}$ the second row to get:

$$\begin{bmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Next use the pivot to get the position above zero. So we multiply 2^{nd} row by 3 and subtract from row 1. ~~to get~~:

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This is the reduced echelon form and the circled positions are the pivots.

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Exercise 8

$$\begin{bmatrix} 1 & 4 & 0 & 7 \\ 2 & 7 & 0 & 10 \end{bmatrix}$$

The first column is pivot column and the top entry 1 in $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is non-zero, so can be used as the pivot. So we multiply 1^{st} row by 2 and subtract from 2^{nd} row.

$$\begin{bmatrix} 1 & 4 & 0 & 7 \\ 0 & -1 & 0 & -4 \end{bmatrix}$$

[We can use this to solve, but let us convert this to reduced echelon form.]

The pivot once you hide the first row is -1, in the second column. Divide 2^{nd} row by -1 to get,

$$\begin{bmatrix} 1 & 4 & 0 & 7 \\ 0 & 1 & 0 & 4 \end{bmatrix}.$$

Now subtract 4×2^{nd} row from 1^{st} row.

$$\begin{bmatrix} 1 & 0 & 0 & -9 \\ 0 & 1 & 0 & 4 \end{bmatrix}.$$

Writing as equations:

$$x_1 \qquad\qquad = -9$$

$x_1 = -1$

$x_2 = 4.$

So x_3 is a free variable.

and so a solution is of the form $(-9, 4, x_3)$, where x_3 can be any number.