Nov 9

Sunday, November 8, 2020 1:30 PM

Sec 5.4

4 $B = \{b_1, b_2, b_3\}$, basis for V, $T: V \to \mathbb{R}^2$
a linear transformation,

$$T(x_1 b_1 + x_2 b_2 + x_3 b_3) = \begin{bmatrix} 2x_1 - 4x_2 + 5x_3 \\ - x_2 + 3x_3 \end{bmatrix}$$

Find the matrix for T relative to B and the standard basis for $\mathbb{R}^2$.

$$M = \begin{bmatrix} 2 & -4 & 5 \\ 0 & -1 & 3 \end{bmatrix}.$$

6 $T: \mathbb{P}_2 \to \mathbb{P}_4$, $T(p(t)) = p(t) + t^2 p(t)$.

a) Find the image of $p(t) = 2 - t + t^2$.

$$(2 - t + t^2) + (2t^2 - t^3 + t^4)$$
$$= 2 - t + 3t^2 - t^3 + t^4.$$

b) Show that T is linear.

$$T(p(t) + q(t)) = (p(t) + q(t)) + t^2(p(t) + q(t))$$
$$= (p(t) + t^2 p(t)) + (q(t) + t^2 q(t))$$
$$= T(p(t)) + T(q(t)).$$

$$T(c\,p(t)) = c\,p(t) + t^2(c\,p(t))$$
$$= c(p(t) + t^2 p(t)) = c\,T(p(t)).$$

c) Find the matrix for T relative to the standard bases for $\mathbb{P}_2$ & $\mathbb{P}_4$.

This will be a 5×3 matrix, since $\dim \mathbb{P}_2 = 3$, $\dim \mathbb{P}_4 = 5$.

$$T(1) = 1 + t^2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \in \mathbb{P}_4.$$

$$T(t) = t + t^3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$T(t^2) = t^2 + t^4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

So the matrix is, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(24) If A and B are square matrices which are similar, rank A = rk B.

$A = [v_1, \ldots, v_n]$ $v_i \in \mathbb{R}^n$. rank A = $\dim \operatorname{Col} A = s$, say. Then we can find $v_{i_1}, \ldots, v_{i_s}$ forming a basis for $\operatorname{Col} A$. Then, $B = P^{-1} A B$, say, one see that $P^{-1} v_{i_1} P$ form a basis for $\operatorname{Col} B$.

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Sec 5.5

(4) Find the (complex) eigenvalues and a basis for each eigen space in $\mathbb{C}^2$. $A = \begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix}$.

The char. poly. is $\lambda^2 - 8\lambda + 17$.

So $\lambda = \dfrac{8 \pm \sqrt{64-68}}{2} = \dfrac{8 \pm 2i}{2}$

$= 4 \pm i$.

$\underline{\lambda = 4+i}$: The eigen space is given by solutions $A\begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix}$. i.e. the null space for,

$\begin{bmatrix} 1-i & -2 \\ 1 & -1-i \end{bmatrix}$. One checks easily $\begin{bmatrix} 2 \\ 1-i \end{bmatrix}$ is a basis for this eigen space. Then its complex conj. $\begin{bmatrix} 2 \\ 1+i \end{bmatrix}$ is a basis for the eigen space corresponding to $4-i$.

(8) $A = \begin{bmatrix} \sqrt{3} & 3 \\ -3 & \sqrt{3} \end{bmatrix}$. This is scaling & rotation. Find the scaling

factor & angle of rotation.

det $A = 12$. So $r = \sqrt{12} = 2\sqrt{3}$.

$$A = \begin{bmatrix} 2\sqrt{3} & 0 \\ 0 & 2\sqrt{3} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}.$$

Angle of rotation φ is, such that

$\cos\varphi = \frac{1}{2}$, $\sin\varphi = -\frac{\sqrt{3}}{2}$.

$\varphi = -\pi/3$. $\lambda = \sqrt{3} \pm 3i$.

(14) Find P so that $P \begin{bmatrix} 5 & -5 \\ 1 & 1 \end{bmatrix} P^{-1} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$.

The char poly of $\begin{bmatrix} 5 & -5 \\ 1 & 1 \end{bmatrix}$ is,

$\lambda^2 - 6\lambda + 10$. So, $\lambda = \frac{6 \pm \sqrt{36 - 40}}{2}$

$= 3 \pm i$. Taking $\lambda = 3 - i$;

a basis for eigen space is,

$\vec{v} = \begin{bmatrix} 5 \\ 2+i \end{bmatrix}$. So $P = \begin{bmatrix} \mathrm{Re}\,\vec{v} & \mathrm{Im}\,\vec{v} \end{bmatrix}$

$= \begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix}$. $C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}$.