Oct 26

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Sec 4.4

(2) $\mathcal{B} = \left\{ \begin{bmatrix} 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 6 \\ 7 \end{bmatrix} \right\}$, $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 8 \\ -5 \end{bmatrix}$

Find $\vec{x}$.

$$\vec{x} = \begin{bmatrix} 4 & 6 \\ 5 & 7 \end{bmatrix} \begin{bmatrix} 8 \\ -5 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}.$$

(14) $\mathcal{B} = \{1 - t^2, t - t^2, 2 - 2t + t^2\}$ is a basis for $\mathbb{P}_2$. Find the co-ordinate vector of $p(t) = 3 + t - 6t^2$ relative to $\mathcal{B}$. So we want to solve.

$$x_1(1 - t^2) + x_2(t - t^2) + x_3(2 - 2t + t^2) = p(t)$$

Equating coeff. of powers of t, we get:

$$\left.\begin{aligned} x_1 + 2x_3 &= 3 \\ x_2 - 2x_3 &= 1 \\ -x_1 - x_2 + x_3 &= -6. \end{aligned}\right| \Rightarrow \begin{bmatrix} x_1 = 7 \\ x_2 = -3 \\ x_3 = -2 \end{bmatrix}.$$

(14)$'$: Why is $\mathcal{B}$ above a basis for $\mathbb{P}_2$?

We write the matrix of $\mathcal{B}$ in the standard basis $\{1, t, t^2\}$:

$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$. Then show that

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ -1 & -1 & 1 \end{bmatrix} \text{ is invertible.}$$

$\begin{bmatrix} 1 & 0 & 2 \end{bmatrix}$ det = 1.

$$A \rightsquigarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 0 & -1 & 3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \to \det = 1.$$

So invertible.

Sec 4.5.

(4) Let H be the subspace of $\mathbb{R}^4$.

$$H = \left\{ \begin{bmatrix} a+b \\ 2a \\ 3a-b \\ -b \end{bmatrix} \;\middle|\; a, b \in \mathbb{R} \right\}.$$

Find a basis and its dimension.

$$\begin{bmatrix} a+b \\ 2a \\ 3a-b \\ -b \end{bmatrix} = a \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix} + b \begin{bmatrix} 1 \\ 0 \\ -1 \\ -1 \end{bmatrix}.$$

So H is spanned by, $u = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}$

and $v = \begin{bmatrix} 1 \\ 0 \\ -1 \\ -1 \end{bmatrix}$. If they are

linearly indep. they will form

a basis for H and $\dim H = 2$.

So we solve $au + bv = \vec{0}$. Second

equation gives $2a = 0$ & 4^{th} gives $-b = 0$.

So $a = b = 0$. Thus they are l. ind.

(12) Find the dimension of the subspace

spanned by $\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} -8 \\ 6 \\ 5 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 7 \end{bmatrix}.$

We look at the matrix:

we look at the matrix:

$$A = \begin{bmatrix} 1 & -3 & -8 & -3 \\ -2 & 4 & 6 & 0 \\ 0 & 1 & 5 & 7 \end{bmatrix}.$$ The subspace

of interest is Col A. So we find the pivot columns of A as usual:

$$A \rightsquigarrow \begin{bmatrix} 1 & -3 & -8 & -3 \\ 0 & -2 & -10 & -6 \\ 0 & 1 & 5 & 7 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -3 & -8 & -3 \\ 0 & -2 & -10 & -6 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

It has 3 pivot cols. So dim = 3.

(32) Let H be a non-zero subspace of V and $T: V \to W$ be a one-to-one linear mapping. Prove that $\dim T(H) = \dim H$.

Pick a basis $u_1, \dots, u_p \in H$ of H, $p = \dim H$.

Then $T(u_1), \dots, T(u_p)$ span $T(H)$. We will show that $T(u_1), \dots, T(u_p)$ are l. ind. Then they will be a basis for $T(H)$ and so $p = \dim T(H)$.

If not, we must have $a_1, \dots, a_p \in \mathbb{R}$, at least one non-zero with,

$$a_1 T(u_1) + \dots + a_p T(u_p) = 0.$$

T is linear, so $T(a_1 u_1 + \dots + a_p u_p)$

$= a_1 T(u_1) + \cdots a_p T(u_p) = 0.$

But T is one-to-one. so $a_1 u_1 + \cdots + a_p u_p = 0.$

$u_1, \ldots, u_p$ is a basis (in particular l. ind)

and so $a_i = 0$ for all i.

Sec 4.6

② ~~Assume~~ $A = \begin{bmatrix} 1 & -3 & 4 & -1 & 9 \\ -2 & 6 & -6 & -1 & -10 \\ -3 & 9 & -6 & -6 & -3 \\ 3 & -9 & 4 & 9 & 0 \end{bmatrix}$ is

row eq. to $B = \begin{bmatrix} 1 & -3 & 0 & 5 & -7 \\ 0 & 0 & 2 & -3 & 8 \\ 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

- Calculate (without any work),

rank A and dim Nul A

B has 3 pivots, 1^{st} col, 3^{rd} col, 5^{th} col.

So rank $A = 3$. Since rk A +

dim Nul $A = 5$, dim Nul $A = 2$.

A basis for Col A: (pivot columns of A)

$$: \left\{ \begin{bmatrix} 1 \\ -2 \\ -3 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ -6 \\ -6 \\ 4 \end{bmatrix}, \begin{bmatrix} 9 \\ -10 \\ -3 \\ 0 \end{bmatrix} \right\}.$$

A basis for Row A. Non-zero rows of

B, (B is in echelon form.)

So 1^{st}, 2^{nd} & 3^{rd} row of B.

For Nul A: We may use B. So

3^{rd} eq. gives $5x_5 = 0$; $x_5 = 0$.

2^{nd} eq gives $2x_3 - 3x_4 + 8x_5 = 0$.

So $x_3 = \frac{3}{2} x_4$.

1^{st} eq.: $x_1 - 3x_2 + 5x_4 - 7x_5 = 0$.

So $x_1 = 3x_2 - 5x_4$. Thus,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 3x_2 - 5x_4 \\ x_2 \\ \frac{3}{2}x_4 \\ x_4 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -5 \\ 0 \\ 3/2 \\ 1 \\ 0 \end{bmatrix}.$$

I will leave you to check

that $\begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} -5 \\ 0 \\ 3/2 \\ 1 \\ 0 \end{bmatrix}$ form a basis for

Nul A.

(3)' Show that if $\vec{u}$, $\vec{v}$ in $\mathbb{R}^n$ then the $n \times n$ matrix $\vec{u}\,\vec{v}^T$ has rk ≤ 1.

The $1 \times n$ matrix $\vec{v}^T$ has rk ≤ 1.

So dim Nul $\vec{v}^T \geq n-1$.

If $\vec{w} \in$ Nul v^T, we have,

$(\vec{u}\,\vec{v}^T)\,\vec{w} = \vec{u}\,(\vec{v}^T\vec{w}) = \vec{u}\;\vec{0} = \vec{0}$.

Thus $\vec{w} \in$ Nul $\vec{u}\,\vec{v}^T$.

Thus Nul $(\vec{u}\,\vec{v}^T) \geq n-1$.

Since $rk(\vec{u}\vec{v}^T) + \dim Nul(\vec{u}\vec{v}^T) = n$,

we get $rk\,\vec{u}\vec{v}^T \le 1$.

Sec 4.7:

(4) Let $\mathcal{A} = \{a_1, a_2, a_3\}$ and $\mathcal{D} = \{d_1, d_2, d_3\}$ be bases for V. Let,

$P = \left[[d_1]_{\mathcal{A}} \ [d_2]_{\mathcal{A}} \ [d_3]_{A} \right]$. Which of the following true?

(i) $[x]_A = P[x]_{\mathcal{D}}$ for all $x \in V$.

True.

(ii) $[x]_{\mathcal{D}} = P[x]_{\mathcal{A}}$ False.

(8) Find the change of co-ordinate matrix from $\mathcal{B}$ to $\mathcal{C}$, where

$\mathcal{B} = \left\{ \begin{bmatrix} -1 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ -5 \end{bmatrix} \right\}$, $\mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$.

$$\underset{\mathcal{C} \leftarrow \mathcal{B}}{P} = \left[\begin{bmatrix} -1 \\ 8 \end{bmatrix}_{\mathcal{C}} \ \begin{bmatrix} 1 \\ -5 \end{bmatrix}_{\mathcal{C}} \right]$$

(There are more than one way to solve this, but let us do this one at a time.)

Recall $\begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ means, Recall $\begin{bmatrix} p \\ q \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ means,

$$\alpha \begin{bmatrix} 1 \\ 4 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} p \\ q \end{bmatrix}.$$

So for $\begin{bmatrix} -1 \\ 8 \end{bmatrix}$, we wish to solve equations, whose augmented matrix is

$$\begin{bmatrix} 1 & 1 & -1 \\ 4 & 1 & 8 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & -1 \\ 0 & -3 & 12 \end{bmatrix}$$

So $\beta = -4$; $\alpha + \beta = -1$, $\alpha = 3$.

$\begin{bmatrix} -1 \\ 8 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$. I will let you calculate

$\begin{bmatrix} 1 \\ -5 \end{bmatrix}_{\mathcal{B}}$.