QUIZ 1

Thursday, September 17, 2020 2:09 PM

1 Solve the system of linear eqns.

$$x + y = 6 \quad \text{--- } (1)$$
$$5x - 4y = 39 \quad \text{--- } (2)$$
$$16x - 11y = 123 \quad \text{--- } (3)$$

Subtract 5 times (1) from (2) & 16 times (1) from (3) to get

$$x + y = 6 \quad \text{--- } (1)$$
$$-9y = 9 \quad \text{--- } (2')$$
$$-27y = 27. \quad \text{--- } (3')$$

Next subtract 3 times (2') from (3'):

$$x + y = 6$$
$$-9y = 9$$
$$0 = 0$$

So from 2^{nd} (3^{rd} gives no information)

$y = -1$. So from 1^{st}, $x - 1 = 6$.

So $\boxed{x = 7, \; y = -1}$

——— x ———

2 Consider a linear system whose

augmented matrix is

$$\begin{bmatrix} 1 & 1 & 3 & 3 \\ 1 & 2 & -5 & 1 \\ 6 & 15 & k & 1 \end{bmatrix}.$$ Find k

so that the system is inconsistent.

Subtract the first row from 2^{nd} and 6 times the 1^{st} row from the 3^{rd} to get:

$$\begin{bmatrix} 1 & 1 & 3 & 3 \\ 0 & 1 & -8 & -2 \\ 0 & 9 & k-18 & -17 \end{bmatrix}$$

Next subtract 9 times the 2^{nd} row from the 3^{rd} to get:

$$\begin{bmatrix} 1 & 1 & 3 & 3 \\ 0 & 1 & -8 & -2 \\ 0 & 0 & k+54 & 1 \end{bmatrix}$$

If $k+54 \neq 0$, the system has 3 pivots and thus a unique solution.

The system is inconsistent

The system is inconsistent if $\boxed{k = -54}$, since then the last eqn is $0 \cdot x_3 = 1$, which has no solution.

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(3) How many pivots does the following matrix have?

$$\begin{bmatrix} 0 & 1 & 0 & -8 \\ 0 & 0 & 1 & 6 \end{bmatrix}$$

It has 2 pivots, the two 1s appearing in the first & second row.

(4) Decide whether the following augmented matrix in echelon form is consistent. If so does it have a unique solution?

$$\begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

The system is inconsistent, since

the 3rd row gives an eqn

$0 \cdot x_3 = -2$, which has no

solution.

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(5) Let $\vec{x}, \vec{y}$ be vectors and $\vec{z} = p\vec{x} + q\vec{y}$

For what values of p, q <u>must</u>

$\text{Span}\{\vec{x}, \vec{z}\} = \text{Span}\{\vec{y}, \vec{z}\}$.

The spans are equal means any linear combination of $\vec{x}, \vec{z}$ is also a linear combination of $\vec{y}, \vec{z}$ and vice versa. It is immediate that this just means $\vec{x}$ (and $\vec{z}$) is a linear combination of $\vec{y}, \vec{z}$ and $\vec{y}$ (and $\vec{z}$) is a linear combination of $\vec{x}, \vec{z}$.

[I have put 'and $\vec{z}$' in paranthesis since $\vec{z}$ is clearly a linear combination of $\vec{y}, z$ as well as $\vec{x}, \vec{z}$, So obvious.]

$\vec{x}$ is a linear combination of $\vec{y}$ and $\vec{z}$, means, we can write

$$\vec{x} = a\vec{y} + b\vec{z} \quad \text{for some } a, b.$$

$$= a\vec{y} + b(p\vec{x} + q\vec{y})$$

$$= bp\vec{x} + (a+bq)\vec{y}.$$

Since $\vec{x}, \vec{y}$ are arbitrary, we see that, one must have $bp = 1$.

In particular, $\boxed{p \neq 0}$

But if $p \neq 0$, then, we can take $b = \frac{1}{p}$ and $a = -bq = \frac{-q}{p}$.

Then $a\vec{y} + b\vec{z} = \vec{x}$.

For the opposite direction, write $\vec{y} = a\vec{x} + b\vec{z} = a\vec{x} + b(p\vec{x} + q\vec{y})$

$$= (a+bp)\vec{x} + bq\vec{y}.$$

As before, we see that $\boxed{q \neq 0}$ is necessary and if $q \neq 0$, we can take $b = \frac{1}{q}$, $a = -\frac{p}{q}$

So the condition is

So the condition is

$$\boxed{p \neq 0 \text{ and } q \neq 0}$$

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(6) $A = \begin{bmatrix} -8 & 3 \\ -8 & -1 \end{bmatrix}$ $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $\vec{b} = \begin{bmatrix} -39 \\ -19 \end{bmatrix}$

Solve for $\vec{x}$, given $A\vec{x} = \vec{b}$.

This is as usual done starting with the augmented matrix,

$$\left[\begin{array}{cc|c} -8 & 3 & -39 \\ -8 & -1 & -19 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c} -8 & 3 & -39 \\ 0 & -4 & 20 \end{array}\right]$$

[we subtracted first row from the second.]

Last equation gives $-4x_2 = 20$ or

$\boxed{x_2 = -5.}$ So the first eqn: gives $-8x_1 - 15 = -39$

or $-8x_1 = -24$. Thus $\boxed{x_1 = 3}$

$$\boxed{\vec{x} = \begin{bmatrix} 3 \\ -5 \end{bmatrix}.}$$

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(7) Solve the eqn $-8x - 2y + 9z = 0$ in parametric form.

[Caution: there are many ways of writing $\vec{u}, \vec{v}$, so if you have a different expression, it might be fine.]

The equation gives,

$9z = 8x + 2y$. So the solutions

are $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ \end{bmatrix}$

$= x \begin{bmatrix} 1 \\ 0 \\ 8/9 \end{bmatrix} + y \begin{bmatrix} \\ \\ \end{bmatrix}.$

So we can take, $\vec{u} = \begin{bmatrix} 1 \\ 0 \\ 8/9 \end{bmatrix}$

and $\vec{v} = \begin{bmatrix} 0 \\ 1 \\ 2/9 \end{bmatrix}.$

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$\lfloor 4/9 \rfloor$ ·

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(8) Consider the following system of linear equations:

$$
\begin{aligned}
2w - x + y - z &= 4 \\
w + x - y + z &= 4 \\
3w &= 8 \\
3w - 3x + 3y - 3z &= 4
\end{aligned}
$$

Write solutions as w, x in terms of y, z.

So the augmented matrix is:

$$
\begin{bmatrix}
2 & -1 & 1 & -1 & 4 \\
1 & 1 & -1 & 1 & 4 \\
3 & 0 & 0 & 0 & 8 \\
3 & -3 & 3 & -3 & 4
\end{bmatrix}
$$

For arithmetic reasons, let us first flip row 1 and 2 to get,

$$
\begin{bmatrix}
1 & 1 & -1 & 1 & 4 \\
2 & -1 & 1 & -1 & 4 \\
3 & 0 & 0 & 0 & 8 \\
3 & -3 & 3 & -3 & 4
\end{bmatrix}
$$

$$
\sim \begin{bmatrix}
1 & 1 & -1 & 1 & 4 \\
0 & -3 & 3 & -3 & -4 \\
\end{bmatrix}
$$

$$\begin{bmatrix} 0 & -3 & 3 & -3 & -4 \\ 0 & -3 & 3 & -3 & -4 \\ 0 & -6 & 6 & -6 & -8 \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} 1 & 1 & -1 & 1 & 4 \\ 0 & -3 & 3 & -3 & -4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

So we get from the second row,

$$-3x = -3y + 3z - 4.$$

or $\boxed{x = y - z + 4/3}$

From 1st eqn:

$$w + (y - z + 4/3) - y + z = 4.$$

So $w = 4 - 4/3 = 8/3$

Thus the solutions are:

$$\boxed{w = 8/3; \quad x = y - z + 4/3}$$

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(9) $\vec{u} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} -1 \\ 6 \\ 3 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 2 \\ -11 \\ 6 \end{bmatrix}$

Decide whether these are linearly independent.

independent.

As usual we write $A = \begin{bmatrix} \vec{u} & \vec{v} & \vec{w} \end{bmatrix}$

$= \begin{bmatrix} 0 & -1 & 2 \\ 1 & 6 & -11 \\ 1 & 3 & 6 \end{bmatrix}$. If this matrix

has 3 pivots, then $\vec{u}, \vec{v}, \vec{w}$

are linearly independent.

$$A \rightsquigarrow \begin{bmatrix} 1 & 3 & 6 \\ 1 & 6 & -11 \\ 0 & -1 & 2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 3 & 6 \\ 0 & 3 & -17 \\ 0 & -1 & 2 \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} 1 & 3 & 6 \\ 0 & 3 & -17 \\ 0 & 0 & 2-\frac{17}{3} \end{bmatrix}.$$

It has 3 pivots, $1, 3, 2-\frac{17}{3}$

So $\vec{u}, \vec{v}, \vec{w}$ are linearly indep.

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(10) $A = \begin{bmatrix} -1 & -3 & 1 \\ 1 & 2 & 0 \\ -1 & 2 & -4 \end{bmatrix}$. Decide

whether $A\vec{x} = \vec{0}$ has a non-zero solution.

We proceed exactly as before.

$$\begin{bmatrix} -1 & -3 & 1 \\ 1 & 2 & 0 \\ -1 & 2 & -4 \end{bmatrix} \rightsquigarrow \begin{bmatrix} -1 & -3 & 1 \\ 0 & -1 & 1 \\ 0 & 5 & -5 \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} -1 & -3 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$ So it has a free variable and thus $A\vec{x} = \vec{0}$ has a non-zero soln.

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(11) $A = \begin{bmatrix} -4 & -4 & 5 \\ 5 & -6 & -4 \\ 6 & -5 & -4 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 21 \\ -2 \\ -11 \end{bmatrix}$

Find a vector $\vec{x}$ such that $A\vec{x} = \vec{b}$.

We have the augmented matrix

$$\begin{bmatrix} -4 & -4 & 5 & 21 \\ 5 & -6 & -4 & -2 \\ 6 & -5 & -4 & -11 \end{bmatrix}$$

$\rightsquigarrow \Big[-4 \quad -4 \quad 5 \quad 21 \Big]$

$$\rightsquigarrow \begin{bmatrix} -4 & -4 & 5 & 21 \\ 0 & -11 & 9/4 & 97/4 \\ 0 & -11 & 14/4 & 41/2 \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} -4 & -4 & 5 & 21 \\ 0 & -11 & 9/4 & 97/4 \\ 0 & 0 & 5/4 & -15/4 \end{bmatrix}$$

So we get $5/4\ x_3 = -15/4$

or $\boxed{x_3 = -3}$

Second equation gives,

$-11 x_2 + 9/4 \times (-3) = 97/4$, which

yields $\boxed{x_2 = -\frac{31}{11}}$

Finally the first eqn is

$-4 x_1 + 4 \times \frac{31}{11} + 5 \times (-3) = 21$

This gives $\boxed{x_1 = \frac{-68}{11}}$

Thus $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -68/11 \\ -31/11 \\ -3 \end{bmatrix}$ is $\rightarrow$

$$\begin{bmatrix} x_3 \end{bmatrix} \quad \begin{bmatrix} -3 \end{bmatrix}$$

the (unique) solution to $T(\vec{x}) = \vec{b}$.

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(12) Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation, $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 \\ x_2 - x_3 \\ x_3 - x_1 \end{bmatrix}$.

Is this onto?

[With some experience you will notice that $(x_1 - x_2) + (x_2 - x_3) + (x_3 - x_1) = 0$, so if $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is in the range of T, $a + b + c = 0$ and thus for example $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is NOT in the range. We will do this formally.]

T is onto means we should be able to solve $T(\vec{x}) = \vec{b}$ for ANY $\vec{b}$. Writing $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \end{bmatrix}$

... $\vec b$. Writing $\vec b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$,

we want to solve $T(\vec x) = \vec b$;

That is $\begin{bmatrix} x_1 - x_2 \\ x_2 - x_3 \\ x_3 - x_1 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$.

The augmented matrix is,

$$\begin{bmatrix} 1 & -1 & 0 & b_1 \\ 0 & 1 & -1 & b_2 \\ -1 & 0 & 1 & b_3 \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} 1 & -1 & 0 & b_1 \\ 0 & 1 & -1 & b_2 \\ 0 & -1 & 1 & b_1 + b_3 \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} 1 & -1 & 0 & b_1 \\ 0 & 1 & -1 & b_2 \\ 0 & 0 & 0 & b_1 + b_2 + b_3 \end{bmatrix}$$

The last eqn says if these are consistent, $0 \cdot x_3 = b_1 + b_2 + b_3$ and so $b_1 + b_2 + b_3 = 0$. Thus for example, if $\vec b = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ the

For example, if $b = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, the system is inconsistent and so T is NOT onto.