See 1.5, 1.7,

Thursday, September 17, 2020 2:09 PM

See 1.5, Exer 8

Easy to reduce the matrix

$$\begin{bmatrix} 1 & -2 & 9 & 5 \\ 0 & 1 & 2 & -6 \end{bmatrix}$$ to reduced

echelon form by adding 2 times the second row to the first. Thus we get

$$A = \begin{bmatrix} 1 & 0 & 13 & -7 \\ 0 & 1 & 2 & -6 \end{bmatrix}.$$

Thus the solutions of $A\vec{x} = \vec{0}$ are:

$$x_1 = -13x_3 + 7x_4$$

$$x_2 = -2x_3 + 6x_4.$$

So the solutions are,

$$\begin{bmatrix} -13x_3 + 7x_4 \\ -2x_3 + 6x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -13 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \vdots \\ x_3 \\ x_4 \end{bmatrix} \qquad \begin{bmatrix} \vdots \\ 0 \end{bmatrix}$$

$$+ x_4 \begin{bmatrix} 7 \\ 6 \\ 0 \\ 1 \end{bmatrix}.$$

Taking,

$$\vec{u} = \begin{bmatrix} -3 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \qquad \vec{v} = \begin{bmatrix} 7 \\ 6 \\ 0 \\ 1 \end{bmatrix},$$

the parametric solutions are

$s\vec{u} + t\vec{v}$, s, v real #s.

—— x ——

<u>Exer 30</u>:

If A is a 3×3 matrix with exactly 2 pivots, does $A\vec{x} = \vec{0}$ have a non-trivial solution?

If A has 2 pivot position, it must have one of the following echelon forms.

$$\begin{bmatrix} * & \square & \square \\ 0 & * & \square \\ 0 & 0 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} * & \square & \square \\ 0 & 0 & * \\ 0 & 0 & 0 \end{bmatrix}.$$

where $* \neq 0$. ($\square$ may or may not be zero. In both cases we see that the solution set has a free variable. (In the first case, x_3 is a free variable and in the second, x_2 is.)

———— x ————

<u>Sec 1.7</u>

Decide whether $\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ -8 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}$ are linearly (in)dependent?

This is same as checking the linear system given by the augmented matrix

$$A = \begin{bmatrix} 0 & 0 & -3 & 0 \\ 0 & 5 & 4 & 0 \\ 2 & -8 & 1 & 0 \end{bmatrix}$$ has a free variable, which we know how to check.

$$A \rightsquigarrow \begin{bmatrix} 2 & -8 & 1 & 0 \\ 0 & 5 & 4 & 0 \\ 0 & 0 & -3 & 0 \end{bmatrix}$$. This

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is already in echelon form and has 3 pivots, 2, 5 & -3.
So it has no free variables and this $A\vec{x} = \vec{0}$ has only trivial solution. This means the 3 vectors are linearly indep.

—— x ——

Exer 14:

Determine h so that

$$\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} -5 \\ 7 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ h \end{bmatrix}$$ are linearly dependent.

As usual we write the coeff. matrix

$$A = \begin{bmatrix} 1 & -5 & 1 \\ -1 & 7 & 1 \\ 3 & 8 & h \end{bmatrix}$$ and see for what values of h, this has less than 3 pivots.

$$A \rightsquigarrow \begin{bmatrix} 1 & -5 & 1 \\ 0 & 2 & 2 \\ 0 & 7 & h-3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -5 & 1 \\ 0 & 1 & 1 \\ 0 & 7 & h-3 \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} 1 & -5 & 1 \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} 1 & -5 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & h-10 \end{bmatrix}.$$

We have 1, 1 and $h-10$ the pivots, unless $h-10=0$. So if $h=10$, A has only 2 pivots and then the 3 vectors would be linearly dependent.

— x —

<u>See 1.8</u> <u>Exer 4</u>.

Let $A = \begin{bmatrix} 1 & -3 & 2 \\ 0 & 1 & -4 \\ 3 & -5 & -9 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 6 \\ -7 \\ -9 \end{bmatrix}$

Define $T(\vec{x}) = A\vec{x}$, $T : \mathbb{R}^3 \to \mathbb{R}^3$

Find a vector $\vec{x}$ such that $T(\vec{x}) = \vec{b}$ and determine whether this $\vec{x}$ is unique.

$$\begin{bmatrix} 1 & -3 & 2 & 6 \\ 0 & 1 & -4 & -7 \\ 3 & -5 & -9 & -9 \end{bmatrix} \rightsquigarrow$$

$$\begin{bmatrix} 1 & -3 & 2 & 6 \\ 0 & 1 & -4 & -7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 & 2 & 6 \\ 0 & 1 & -4 & -7 \\ 0 & 4 & -15 & -27 \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} 1 & -3 & 2 & 6 \\ 0 & 1 & -4 & -7 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Last eqn says $x_3 = 1$. Second eqn. is then $x_2 - 4 = -7$, so $x_2 = -3$

1st eqn is then,

$$x_1 + 9 + 2 = 6, \quad \text{so, } x_1 = -5$$

Thus $\begin{bmatrix} -5 \\ -3 \\ 1 \end{bmatrix}$ is the unique solution.

———— x ————

Exer 20

Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\vec{v}_1 = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 7 \\ -3 \end{bmatrix}$

$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $T(\vec{x}) = x_1 \vec{v}_1 + x_2 \vec{v}_2$.

Find a matrix A such that

$T(\vec{x}) = A\vec{x}$, for each $\vec{x}$.

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then,

$$A\vec{x} = \begin{bmatrix} a x_1 + b x_2 \\ c\, x_1 + d x_2 \end{bmatrix}.$$

$$A\vec{x} = \begin{bmatrix} a x_1 + b x_2 \\ c\, x_1 + d x_2 \end{bmatrix}.$$

$$T(\vec{x}) = x_1 \vec{v}_1 + x_2 \vec{v}_2 = \begin{bmatrix} -2x_1 + 7x_2 \\ 5x_1 - 3x_2 \end{bmatrix}.$$

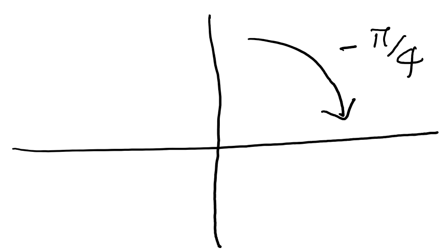
$T(\vec{x}) = A\vec{x}$ means, A must be,

$$\begin{bmatrix} -2 & 7 \\ 5 & -3 \end{bmatrix}.$$

———— x ————

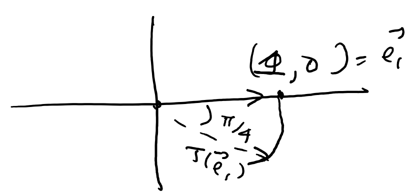
See 1.9 Exer 4

Find the standard matrix for the rotation of $\mathbb{R}^2$ by an angle $-\pi/4$.

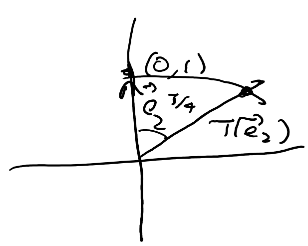


We calculate $T(\vec{e}_1)$ & $T(\vec{e}_2)$.

$$T(\vec{e}_1) = \begin{bmatrix} \cos \pi/4 \\ -\sin \pi/4 \end{bmatrix}$$



$$= \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}.$$



$$T(\vec{e}_2) = \begin{bmatrix} \sin \pi/4 \\ \cos \pi/4 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

Thus A

Thus $A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$.

—— x ——

Exer 22:

Let $T: \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation such that,

$T(x_1, x_2) = (x_1 - 2x_2, -x_1 + 3x_2, 3x_1 - 2x_2)$

Find $\vec{x}$ s.t. $T(\vec{x}) = (-1, 4, 9)$.

The matrix for T is,

$\begin{bmatrix} 1 & -2 \\ -1 & 3 \\ 3 & -2 \end{bmatrix}$. So the linear system has the augmented matrix,

$$\begin{bmatrix} 1 & -2 & -1 \\ -1 & 3 & 4 \\ 3 & -2 & 9 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 3 \\ 0 & 4 & 12 \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}.$$

So $x_2 = 3$ (2^{nd} eqn)

$x_1 - 6 = -1$ (1^{st} eqn)

So $x_1 = 5$. So $\vec{x} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$.

—— x ——