Answers to Homework 1, Math 310

We will assume that our present knowledge is just the following.

We have the natural numbers or counting numbers, usually denoted by the letter \( \mathbb{N} \). These are just the collection \{1, 2, 3, \ldots \}. These have the following basic properties. Lower case English letters will denote natural numbers in the following.

- **closure**: We may add two natural numbers to get a natural number. Addition is denoted by the symbol \(+\). Similarly we can multiply two natural numbers to get a natural number. Multiplication is denoted by either \( \cdot \) or just writing the numbers next to each other.
- **commutativity**: \( a + b = b + a \) and \( ab = ba \) for all natural numbers \( a, b \).
- **associativity**: \( a + (b + c) = (a + b) + c \) and \( a(bc) = (ab)c \) for all natural numbers \( a, b, c \).
- **distributivity**: \( a(b + c) = ab + ac \) for all natural numbers \( a, b, c \).
- **Notation**: In this notation, we have \( 2 = 1 + 1 \), \( 3 = 2 + 1 \) etc. and all these are different.

(1) We start with a definition.

**Definition 1.** A natural number \( a \) is called even if there exists another natural number \( b \) such that \( a = 2b \). A natural number \( a \) is called odd if \( a + 1 \) is even.

Write a know-show table and a proof for the following two theorems.

**Theorem 1.** If \( a \) is even, then \( a + 2 \) is even, where \( a \) is a natural number.

**Theorem 2.** If \( a \) is even, then \( 3a^2 + 4a + 7 \) is odd where \( a \) is a natural number and the notation \( a^2 \) as usual stands for \( a \cdot a \).

The Know-show table for Theorem 1.

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<th>Reason</th>
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<td>Hypothesis</td>
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<td>2</td>
<td>( a = 2b ) for some natural number ( b )</td>
<td>Definition 1</td>
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<td>3</td>
<td>( a + 2 = 2b + 2 )</td>
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<td>4</td>
<td>( a + 2 = 2(b + 1) )</td>
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<td>( b + 1 ) is a natural number</td>
<td>Closure</td>
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<tr>
<td>6</td>
<td>( a + 2 ) is even</td>
<td>Definition 1</td>
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Next we write a proof in English.

**Proof.** Since \(a\) is even, by definition 1, we have a natural number \(b\) such that \(a = 2b\). Then,

\[
    a + 2 = 2b + 2 = 2(b + 1)
\]

the last equality by distributivity. Since \(b + 1\) is a natural number by closure, we see that by definition 1 that \(a + 2\) is even from equation 1 above. \(\square\)

For the second theorem, I will write a proof in English and not the Know-show table.

**Proof.** By definition 1, since \(a\) is even, we have a natural number \(b\) such that,

\[
    a = 2b.
\]

We wish to show that \(3a^2 + 4a + 7\) is odd. By definition 1, we must show that \((3a^2 + 4a + 7) + 1\) is even.

\[
    (3a^2 + 4a + 7) + 1 = 3a^2 + 4a + 8 \quad \text{algebra}
\]

\[
    = 3(2b)^2 + 4(2b) + 8 \quad \text{substitution from (2)}
\]

\[
    = 12b^2 + 8b + 8 \quad \text{algebra}
\]

\[
    = 2(6b^2 + 4b + 4) \quad \text{algebra}
\]

Since \(6b^2 + 4b + 4\) is a natural number by closure, by definition 1 we see that \((3a^2 + 4a + 7)\) is odd.

(Here we have abbreviated by the word *algebra* various allowed operations from the properties of numbers, without explicitly writing them. If you are uncomfortable, you should write them out in several steps with appropriate justifications. In fact, I would expect this when you submit homework, though I allow myself this freedom). \(\square\)

(2) With our present knowledge, can you prove that any natural number is either even or odd? If you can, write a proof. If you cannot, explain why.

No, I do not know how to prove this with our present knowledge. Almost all such statements need Mathematical Induction. So, I state this very important property next, of our natural numbers.

(a) For two natural numbers \(a, b\), if \(a + 1 = b + 1\), then \(a = b\).
(b) (Mathematical induction) Let $S$ be a collection of natural numbers. Assume $S$ has the property that,

(i) $1 \in S$.

(ii) If $a \in S$ then $a + 1 \in S$.

Then $S = \mathbb{N}$.

**Theorem 3.** Any natural number is either odd or even.

**Proof.** Let $S$ be the collection of natural numbers which are either even or odd. First we check that $1 \in S$. Since $1 + 1 = 2 = 2 \cdot 1$, we see that $1$ is odd by definition 1. So, $1 \in S$. Next we check that if $a \in S$, then $a + 1 \in S$. Since $a \in S$, it must be odd or even. Let us look at the two cases.

If $a$ is even, then $a = 2b$ for some natural number by definition 1. Thus $a + 1 = 2b + 1$. Then $(a + 1) + 1 = (2b + 1) + 1 = 2b + 2 = 2(b + 1)$ and since $b + 1$ is a natural number by closure, we see that $a + 1$ is odd by definition 1. So, $a + 1 \in S$.

If $a$ is odd, then by definition 1, $a + 1$ is even and thus $a + 1 \in S$. Thus by induction we see that $S = \mathbb{N}$. That is, any natural number is either even or odd. This proves the theorem.

□

(3)

**Definition 2.** If $a, b$ are natural numbers we say $a$ is greater than $b$, written $a > b$, if there exists a natural number $k$ with $a = b + k$.

Write a know-show table and a proof for the following theorem.

**Theorem 4.** If $a, b$ are natural numbers and $a > b$, then $a^2 + b^2 > 2ab$.

Again we will only write a proof.

**Proof.** Since $a > b$, by definition 2, we can write $a = b + k$ for some natural number $k$. Then,

\[
\begin{align*}
a^2 + b^2 &= (b + k)^2 + b^2 & \text{substitution} \\
&= b^2 + 2bk + k^2 + b^2 & \text{algebra} \\
&= 2b^2 + 2bk + k^2 & \text{algebra} \\
&= 2b(b + k) + k^2 & \text{distributivity} \\
&= 2ab + k^2 & \text{substitution}
\end{align*}
\]

Thus we see that $a^2 + b^2 = 2ab + k^2$ and since $k^2$ is a natural number, by definition 2, we see that $a^2 + b^2 > 2ab$. □
(4) With our present knowledge, can you prove that if $a, b$ are any two natural numbers, then exactly one of the following must occur:

$$a = b \quad \text{or} \quad a > b \quad \text{or} \quad b > a?$$

If you can, write a proof and if you can not, explain why.

No, without Induction I do not know how to prove this. I urge all of you to try to do it using induction.