Solutions 12, Math 310

(1) Show that if $S, T \subset \mathbb{R}$ are bounded sets then so is $S \cup T$.

Proof. This is trivial. $S$ is bounded means there exists $A, B \in \mathbb{R}$ such that for any $s \in S$, $A \leq s \leq B$. Similarly, there exists $C, D \in \mathbb{R}$ so that for any $t \in T$, $C \leq t \leq D$. If $M = \min\{A, C\}$ and $N = \max\{B, D\}$, one easily sees that for any $z \in S \cup T$, $M \leq z \leq N$, proving that $S \cup T$ is bounded. □

(2) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and as usual, for real numbers $a \leq b$ we denote by $[a, b]$, the closed interval consisting of real numbers $x$ such that $a \leq x \leq b$.

(a) Show that for $a < b$, real numbers, $f([a, b]) = \{f(x) \mid x \in [a, b]\}$ is a bounded set. (Hint: Consider the set $S = \{x \in [a, b] \mid f([a, x])$ is bounded$\}$).

Proof. We consider $S$ as described in the hint. Clearly $a \in S$ and $S \subset [a, b]$. So, $S$ is a non-empty bounded set and thus by the theorem proved in class, has a supremum, say $t$. First we show that $t = b$.

If $t \neq b$, then since $b$ is an upper bound for $S$, we must have $t < b$. Since $f$ is continuous at $t$, there exists a $\delta > 0$ so that for all $x$ with $|x - t| < \delta$, we have $|f(x) - f(t)| < 1$. We may clearly assume that $\delta < b - t$. Thus, if $x \in [t - \delta/2, t + \delta/2]$, $f(t) - 1 \leq f(x) \leq f(t) + 1$ and so $f([t - \delta/2, t + \delta/2])$ is bounded. Since $t$ is the supremum for $S$, there exists an element $s \in S$ such that $t - \delta/2 < s \leq t$. So, by definition of $S$, we know that $f([a, s])$ is bounded. Since $[a, t + \delta/2] = [a, t] \cup [t - \delta/2, t + \delta/2]$, we see from the first part of the problem, that $f([a, t + \delta/2])$ is bounded and thus $t + \delta/2 \in S$, contradiciting the assumption that $t$ is the supremum for $S$. So we must have $t = b$.

Exactly as in the previous paragraph, one can see that $f([a, b])$ is bounded once we have $t = b$. □

(b) Show that $f([a, b]) = [c, d]$ for some real numbers $c \leq d$.

(Do not forget the intermediate value theorem).

Proof. From the previous part, we know that $f([a, b])$ is bounded and clearly non-empty. So, we have $c$ the infimum and $d$ the supremum of this set. Thus, $f([a, b]) \subset [c, d]$. So, if we show that both $c$ and $d$ are in $f([a, b])$, then by intermediate value theorem, we must have $[c, d] \subset f([a, b])$.
which will prove what we set out to prove. We will show that \(c \in f([a, b])\), the other part being similar.

Since \(c\) is the infimum for the set \(f([a, b])\), for any \(n \in \mathbb{N}\), there exists an \(x_n \in [a, b]\) so that \(c \leq f(x_n) < c + \frac{1}{n}\). Let us look at two cases, when the set \(T = \{x_1, x_2, \ldots\}\) is finite or infinite. In the first case, if none of these \(f(x_n) = c\), then let \(r = \min\{f(x_n) - c\}\), which makes sense, since there are only finitely many distinct \(x_n\)s and all these \(f(x_n) - c > 0\) and hence \(r > 0\). But, if we take an \(n \in \mathbb{N}\) so that \(1/n < r\), we see that \(f(x_n) - c < 1/n < r\), which is not possible by choice of \(r\). So, \(f(x_n) = c\) for some \(n\) and since \(x_n \in [a, b]\), we are done.

Now, let us assume that the above set \(T\) is infinite. Since \(T\) is bounded, there exists an infinite subset say \(\{y_1, y_2, \ldots\} \subset T\), which is a CS, from an earlier exercise. I claim that if we take \(y = \lim y_n\), then \(y \in [a, b]\) and \(f(y) = c\), which will prove what we need to prove. Since \(a \leq y_n \leq b\) for all \(n\), we have \(a \leq y \leq b\) and thus \(y \in [a, b]\).

To show that \(f(y) = c\), we proceed as follows. Since \(f(y) = \lim f(y_n)\), suffices to show that given \(\epsilon > 0\), there exists an \(N \in \mathbb{N}\) such that for all \(n \geq N\), \(|f(y_n) - c| < \epsilon\). Choose an \(m \in \mathbb{N}\) so that \(1/m < \epsilon\). Since \(\{x_1, \ldots, x_{m-1}\}\) is a finite set, there are only finitely many \(y_n\)s in this set. So, there exists an \(N\) such that for all \(n \geq N\), \(y_n\) is not in this set. That is, for all such \(y_n\)s, \(y_n = x_l\) with \(l \geq m\). Thus for such an \(n \geq N\), we see that \(|f(y_n) - c| = |f(x_l) - c| < 1/m < \epsilon\). This proves that \(\lim f(y_n) = c\), which proves the result.

(3) Show that there are infinitely many points \((\alpha, \beta)\) on the unit circle \(x^2 + y^2 = 1\) with \(\alpha, \beta \in \mathbb{Q}\). (Pythagorean triples).

Proof. If \((x, y, z)\) is a Pythagorean triple (that is \(x, y, z \in \mathbb{N}\) and \(x^2 + y^2 = z^2\)), then clearly, \((x/z)^2 + (y/z)^2 = 1\). I will leave you to check that there infinitely many such using the fact that there are infinitely many primitive Pythagorean triples.

(4) Consider the equation \(x^2 - 2y^2 = 1\), called the Pell’s equation. We wish to find integer solutions to this equation. It is clear that \(x = 3, y = 2\) is such a solution.

(a) Denote by \(\mathbb{Z}[\sqrt{2}]\), the set of real numbers of the form \(a + b\sqrt{2}\) where \(a, b \in \mathbb{Z}\). Show that this set is closed under addition and multiplication.

Proof. This is straight forward.
(b) If \( a, b \in \mathbb{Z} \) with \((a + b\sqrt{2})(c + d\sqrt{2}) = 1\), where \( c, d \in \mathbb{Z} \), show that \( c = a, d = -b \) or \( c = -a, d = b \).

Proof. The above gives two equations, \( ac + 2bd = 1 \) and \( ad + bc = 0 \). The first implies that \( \gcd(a, b) = 1 \). From the second, then we see that since \( a \) divides \( bc \) and \( \gcd(a, b) = 1 \) and so \( a \) divides \( c \). Write \( c = qa \) for some \( q \in \mathbb{Z} \). The second immediately implies that \( a = 0 \) or \( d = -qb \). But \( a \neq 0 \) since then we get the absurd equation \( 2bd = 1 \), an even number equals an odd number. Thus \( d = -qb \). Substituting in the first, we get \( q(a^2 - 2b^2) = 1 \). Then \( q = \pm 1 \) and this proves the result.

(c) If \( a, b \in \mathbb{Z} \) and \((a + b\sqrt{2})(3 + 2\sqrt{2}) = c + d\sqrt{2}\), show that \( c^2 - 2d^2 = a^2 - 2b^2 \).

Proof. By multiplying, it is clear that \( c = 3a + 4b, d = 3b + 2a \). Thus,
\[
c^2 - 2d^2 = (3a + 4b)^2 - 2(3b + 2a)^2 \\
= (9a^2 + 24ab + 16b^2) - 2(9b^2 + 12ab + 4a^2) \\
= a^2 - 2b^2
\]

(d) If we write \((3 + 2\sqrt{2})^n = a_n + b_n\sqrt{2}\) with \( a_n, b_n \in \mathbb{Z} \), show that \( a_n^2 - 2b_n^2 = 1 \).

Proof. Proof is by induction on \( n \), \( n = 1 \) case being trivial. So, we assume the result is true for \( n \) and prove the result for \( n + 1 \). But, \( a_{n+1} + b_{n+1}\sqrt{2} = (a_n + b_n\sqrt{2})(3 + 2\sqrt{2}) \) and from the previous part, we know that \( a_{n+1}^2 - 2b_{n+1}^2 = a_n^2 - 2b_n^2 = 1 \).