Solutions to Homework 6, Math 310

(1) As usual, define $n! = 1 \cdot 2 \cdot 3 \cdots n$ where $n$ is any natural number. By convention $0!$ is defined to be $1$. Similarly $\binom{n}{r}$ for $0 \leq r \leq n$ is defined to be,

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

(a) Show that

$$\binom{n+1}{r+1} = \binom{n}{r} + \binom{n}{r+1}$$

Proof. We give a direct proof using the definition.

$$\binom{n}{r} + \binom{n}{r+1} = \frac{n!}{r!(n-r)!} + \frac{n!}{(r+1)!(n-r-1)!}$$

$$= \frac{n!}{(r+1)!(n-r)!} (r+1+n-r)$$

$$= \frac{n!}{(r+1)!(n-r)!} (n+1)$$

$$= \frac{(n+1)!}{(r+1)!(n-r)!}$$

$$= \binom{n+1}{r+1} \quad \square$$

(b) Prove the binomial theorem,

$$(x+y)^n = x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-1}xy^{n-1} + y^n$$

Proof. We prove this by induction on $n$. So, let $P(n)$ be the statement given by the formula above. Then the statement $P(1)$ just becomes

$$x+y = x+y,$$

which is clearly true. To prove $P(n) \Rightarrow P(n+1)$, we start with the statement of $P(n)$ which is assumed to be true and multiply both sides of the equation by $x+y$ to get,

$$(x+y)^{n+1} = (x+y)(x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-1}xy^{n-1} + y^n)$$

1
Now use distributive law to write the right hand side as,

\[(x + y)^{n+1} = x^{n+1} + \binom{n}{1} x^n y + \binom{n}{2} x^{n-1} y^2 + \cdots\]

\[= x^n y + \binom{n}{1} x^{n-1} y^2 + \binom{n}{2} x^{n-2} y^3 + \cdots\]

This makes the right hand side to be,

\[x^{n+1} + \left( \binom{n}{1} + \binom{n}{0} \right) x^n y + \left( \binom{n}{2} + \binom{n}{1} \right) x^{n-1} y^2 + \cdots\]

\[+ \left( \binom{n}{n} + \binom{n}{n-1} \right) x y^n + y^{n+1}\]

Now using the previous problem one has \(\binom{n}{1} + \binom{n}{0} = \binom{n+1}{1}\),
\(\binom{n}{2} + \binom{n}{1} = \binom{n+1}{2}\), \ldots , \(\binom{n}{n} + \binom{n}{n-1} = \binom{n+1}{n}\). Substituting these, we get,

\[(x + y)^{n+1} = x^{n+1} + \binom{n+1}{1} x^n y + \binom{n+1}{2} x^{n-1} y^2 + \cdots ,\]

which is precisely \(P(n + 1)\).

\[\square\]

(2) Let \(\mathbb{Z}\), the set of integers be the universal set. Write using roster method the following sets.

(a) \(A = \{a \in \mathbb{Z} \mid a^2 < 7\}\).
\(A = \{-2, -1, 0, 1, 2\}\)

(b) \(B = \{a \in \mathbb{Z} \mid \exists b \in \mathbb{Z}, ab = 1\}\).
\(B = \{-1, 1\}\)

(c) \(C = \{a \in \mathbb{Z} \mid a = 3b + 1, b \in \mathbb{Z}\}\).
\(C = \{\ldots, -5, -2, 1, 4, 7, \ldots\}\)

(3) Let the notation be as in the previous problem. Write using roster method the following sets.

(a) \(B \cup C = \{\ldots, -8, -5, -2, -1, 1, 4, 7, 10, \ldots\}\)
(b) \(A - C = \{-1, 0, 2\}\)
(c) \(A \cap B \cap C = \{1\}\)

(4) Prove that for any three sets \(P, Q, R \subseteq \Omega\), where \(\Omega\) is the universal set, \(P \cap (Q \cup R) = (P \cap Q) \cup (P \cap R)\) and \((P \cup Q)^c = P^c \cap Q^c\).

Proof. We will only prove part of the first statement, the others being similar.

As usual, we show that \(P \cap (Q \cup R) \subseteq (P \cap Q) \cup (P \cap R)\) and the reverse inclusion. So, let \(a \in P \cap (Q \cup R)\). Then by definition of intersection, we have \(a \in P\) and \(a \in Q \cup R\). By definition of
union, we have therefore $a \in P$ and $(a \in Q \text{ or } a \in R)$. Thus we have $(a \in P \text{ and } a \in Q)$ or $(a \in P \text{ and } a \in R)$. So, $a \in P \cap Q$ or $a \in P \cap R$. This implies $a \in (P \cap Q) \cup (P \cap R)$. This proves one inclusion.

The rest I leave you to check. □

(5) Let $S_1, S_2, \ldots, S_n$ be finite sets.

(a) Show that $\bigcup_{i=1}^{n} S_i$ is a finite set, where the above notation is the short form for $S_1 \cup S_2 \cup \cdots \cup S_n$.

Proof. We prove this by induction. So, let $P(n)$ be the statement that for any collection $S_1, \ldots, S_n$ of finite subsets, $\bigcup_{i=1}^{n} S_i$ is a finite set. Clearly $P(1)$ is true, which just says that any finite set is a finite set.

Next we show that $P(n) \Rightarrow P(n + 1)$. Let $T = \bigcup_{i=1}^{n} S_i$. Then it is clear that $\bigcup_{i=1}^{n+1} S_i$, which we wish to prove is a finite set is just $T \cup S_{n+1}$. By $P(n)$, we know that $T$ is finite. So, we only need to show that the union of two finite sets is finite. This was done in class. □

(b) Show that $|\bigcup_{i=1}^{n} S_i| \leq \sum_{i=1}^{n} |S_i|$.

Proof. This can be easily proved using the roster method. But, let me prove it using induction. So, let $P(n)$ be just the formula above (which depends on $n$, as the number of sets, $S_i$). As usual, we check first that $P(1)$ is true. $P(1)$ is just the statement $|S_1| \leq |S_1|$, which is clearly true.

Next we check that $P(n) \Rightarrow P(n + 1)$. Given finite sets $S_i, 1 \leq i \leq n + 1$, consider $T = \bigcup_{i=1}^{n} S_i$. Then, $\bigcup_{i=1}^{n+1} S_i = T \cup S_{n+1}$. Thus, from what we saw in class, $|\bigcup_{i=1}^{n+1} S_i| \leq |T| + |S_{n+1}|$. Using the fact that $P(n)$ is true, we see that $|T| \leq \sum_{i=1}^{n} |S_i|$. Putting these together, we have,

$$|\bigcup_{i=1}^{n+1} S_i| \leq |T| + |S_{n+1}| \leq \sum_{i=1}^{n} |S_i| + |S_{n+1}| = \sum_{i=1}^{n+1} |S_i|$$

Thus $P(n) \Rightarrow P(n + 1)$ and by induction $P(n)$ is true for all $n \in \mathbb{N}$. This proves the above formula for all $n \in \mathbb{N}$. □
(c) If $S_i \cap S_j = \emptyset$ for all $i \neq j$, show that $|\bigcup_{i=1}^n S_i| = \sum_{i=1}^n |S_i|$. 

Proof. This too can easily proved without induction, but we will give a proof using induction. As before, let $P(n)$ be the statement as in the formula above. That is $P(n)$ is the statement, that for finite sets $S_1, \ldots, S_n$ with $S_i \cap S_j = \emptyset$ for all $i \neq j$, $|\bigcup_{i=1}^n S_i| = \sum_{i=1}^n |S_i|$. Then $P(1)$ is just the statement $|S_1| = |S_1|$, which is obviously true. Next, we shall prove that $P(n) \Rightarrow P(n+1)$. As before, given finite sets $S_1, \ldots, S_{n+1}$, with $S_i \cap S_j = \emptyset$, let $T = \bigcup_{i=1}^n S_i$. Then we have $\bigcup_{i=1}^{n+1} S_i = T \cup S_{n+1}$. On the other hand, by distributivity, 

$$T \cap S_{n+1} = (\bigcup_{i=1}^n S_i) \cap S_{n+1} = \bigcup_{i=1}^n (S_i \cap S_{n+1}) = \emptyset,$$

since $S_i \cap S_{n+1} = \emptyset$ for all $i < n + 1$. Thus from what we saw in class,

$$|\bigcup_{i=1}^{n+1} S_i| = |T \cup S_{n+1}| = |T| + |S_{n+1}|. \quad (1)$$

Since $P(n)$ is assumed to be true, we have,

$$|T| = |\bigcup_{i=1}^n S_i| = \sum_{i=1}^n |S_i|, \quad (2)$$

since $S_i \cap S_j = \emptyset$ for all $i \neq j$. Substituting equation 2 in 1, we get,

$$|\bigcup_{i=1}^{n+1} S_i| = \sum_{i=1}^{n+1} |S_i|,$$

proving $P(n+1)$ is true. Thus by induction, $P(n)$ is true for all $n$ and thus the formula above is true. 

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